Nordic Particle Accelerator School
August 17-23, 2015, Lund, Sweden

## Problem Set 1: Introduction to Accelerator Physics

Monday, August 17, 2015

## Problem 1

a) How much faster is an electron with $\gamma=10$ than an electron with $\gamma=5$ ?
b) What is the energy (in MeV ) of a proton with $\gamma=10$ ? What about an electron?

## Solution

a) Lorentz factor $\gamma$ is

$$
\gamma=\frac{1}{\sqrt{1-\beta^{2}}}
$$

where $\beta=v / c$. Then

$$
\begin{aligned}
& \frac{1}{\gamma^{2}}=1-\beta^{2} \\
& \beta^{2}=1-\frac{1}{\gamma^{2}} \\
& \frac{v_{2}}{v_{1}}=\frac{\beta_{2}}{\beta_{1}}=\sqrt{\frac{1-\frac{1}{10^{2}}}{1-\frac{1}{5^{2}}}}=1.0155
\end{aligned}
$$

b) The total energy $E$ is

$$
E=m c^{2}
$$

where $m$ is the relativistic mass of a particle dependent on $\gamma$ and a rest mass $m_{0}: m=\gamma m_{0}$. Hence, for $\gamma=10$

$$
E=\gamma m_{0} c^{2}= \begin{cases}9390 \mathrm{MeV} & \text { for } p \\ 5.11 \mathrm{MeV} & \text { for } e^{-}\end{cases}
$$

## Problem 2

What is the length of the $1^{\text {st }}$ and $5^{\text {th }}$ drift section in Widerøe linear accelerator with $f_{R F}=7 \mathrm{MHz}$, energy gain in the gap 1 MeV and starting kinetic energy of 100 keV . Calculate for protons and electrons. Assume the accelerating gaps to be very short compared to the drift tubes.


Scheme of a Widerøe linear accelerator.

## Solution

The distance traveled by a particle is $d=t v$. Let's call the length of the $1^{s t}$ drift $d_{1}$ and the $5^{t h}$ drift $d_{5}$. Condition for acceleration is synchronization of particle motion with RF frequency:

$$
t=\frac{1}{2} \frac{1}{f_{R F}}
$$

The velocity of a particle is

$$
\begin{aligned}
\qquad v & =\beta c=c \sqrt{1-\frac{1}{\gamma^{2}}}, \\
\text { where } & \gamma=\frac{E}{m_{0} c^{2}}, \\
\text { and the total energy is } & E=E_{0}+\Delta E .
\end{aligned}
$$

Then, using the above, the length of the drift section is

$$
d=t v=\frac{1}{2} \frac{1}{f_{R F}} \beta c=\frac{1}{2} \frac{1}{f_{R F}} c \sqrt{1-\frac{1}{\gamma^{2}}}=\frac{1}{2} \frac{c}{f_{R F}} \sqrt{1-\left(\frac{m_{0} c^{2}}{E}\right)^{2}} .
$$

For protons:
$m_{0} c^{2}=938 \mathrm{MeV}$
$\Delta E_{1}=1.1 \mathrm{MeV}$
$\Delta E_{5}=5.1 \mathrm{MeV}$
then, $d_{1}=1.04 \mathrm{~m}$ and $d_{5}=2.26 \mathrm{~m}$.

## For electrons:

$$
\begin{aligned}
& m_{0} c^{2}=0.511 \mathrm{MeV} \\
& \Delta E_{1}=1.1 \mathrm{MeV} \\
& \Delta E_{5}=5.1 \mathrm{MeV}
\end{aligned}
$$

then, $d_{1}=20.3 \mathrm{~m}$ and $d_{5}=21.3 \mathrm{~m}$.

Electrons become relativistic and, thus, with energy gain there is no big increase in velocity.

## Problem 3

How large is a microtron with $B=1.2 \mathrm{~T}$ and $E_{\text {max }}=25 \mathrm{MeV}$ ?

## Solution

In a uniform magnetic field perpendicular to the velocity vector the charged particle will move in a circle of constant radius $r$ (also called gyroradius). The Lorentz force in this case is equal to the centripetal force:


Scheme of a microtron.

$$
\frac{m v^{2}}{r}=q v B
$$

Then the radius of a microtron is $\quad r=\frac{m v}{q B}=\frac{p}{q B}$.
Using the energy-momentum relation ${ }^{1} E^{2}=p^{2} c^{2}+m_{0}^{2} c^{4}$ we can express the momentum $p$ and find $r$

$$
r=\frac{p}{q B}=\frac{\sqrt{E^{2}-m_{0}^{2} c^{4}}}{c q B}
$$

For the above case $m_{0}^{2} c^{4} \ll E^{2}$ and the momentum can be approximated as $p \approx \frac{E}{c}$. Then

$$
r=\frac{E}{c q B}=0.07 \mathrm{~m}
$$

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\({ }^{1}\) You should be able to derive the energy-momentum relation if you don't remember it by heart:
    \(p=m v=\gamma \beta m_{0} c \quad \mid \cdot c\)
    \(\left.p c=\gamma \beta m_{0} c^{2}=\gamma \sqrt{1-\frac{1}{\gamma^{2}}} m_{0} c^{2}=\sqrt{\gamma^{2}-1} m_{0} c^{2} \quad \right\rvert\,()^{2}\)
    \(p^{2} c^{2}=\left(\gamma^{2}-1\right) m_{0}^{2} c^{4}=E^{2}-m_{0}^{2} c^{4}\)
    \(E^{2}=p^{2} c^{2}+m_{0}^{2} c^{4}\)
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## Problem 4

ESRF synchrotron is 844.4 m in circumference. It has $\mathrm{n}=64$ dipole magnets, each $L=2.45 \mathrm{~m}$ long, and operates at 6 GeV energy. Calculate the bending radius of dipoles $r$.

## Solution

The full turn is $2 \pi$, then, the bending angle of a one dipole is

$$
\theta=\frac{2 \pi}{n}
$$

Since $L \ll r$, the small angle approximation gives $\theta=L / r$, and the bending radius is

$$
r=\frac{L}{\theta}=\frac{L n}{2 \pi} \approx 25 \mathrm{~m} .
$$

## Problem 5

A dipole magnet has a 1-turn coil with 400 A .
a) How large should the gap be to produce 0.1 T ?
b) How large is the power consumption if you use a copper wire ( 1 m long) with an area $S=5 \mathrm{~mm}^{2}$ and a resistivity $\rho=1.7 \times$ $10^{-2} \Omega \mathrm{~mm}^{2} / \mathrm{m}$ ?
c) Change to a 20 turn coil using 20 times longer wire. At 0.1 T how large is the power?


Schematics of a dipole.

## Solution

a) The magnetic field in a dipole is

$$
B=\frac{n I \mu_{0}}{h}
$$

where $n$ is number of turns in the coil, $I$ is the current in the coil, $h$ - the gap, and $\mu_{0}=$ $4 \pi \times 10^{-7} \mathrm{NA}^{-2}$ magnetic constant. Then, the gap is

$$
h=\frac{n I \mu_{0}}{B}=5 \mathrm{~mm}
$$

b) The power is

$$
P=R I^{2}=\rho \frac{l}{S} I^{2}=544 \mathrm{~W}
$$

c) With 20 windings in the coil resistance becomes $R=\rho \frac{20 l}{S}$ and using the expression from part $b$ ) we find $P=27 \mathrm{~W}$.

## Problem 6

A quadrupole has a strength of $20 \mathrm{~T} / \mathrm{m}$ and a pole radius of 20 mm . How far from the center do you have to go to find the poles only 1 mm apart?

## Solution

The potential giving the surface of a quadrupole is

$$
\begin{aligned}
V & =g x z \\
\text { that is } \quad x z & =\frac{V}{g}=\text { const } .
\end{aligned}
$$



The magnet pole arrangement for a quadrupole magnet.

We can find this constant if we know $x$ and $z$ at a certain point on the pole (and we do since we know that the closest point is 20 mm away and it is on the diagonal $(\mathrm{x}=\mathrm{z})$ because of the symmetry of a quadrupole magnet).

$$
\begin{aligned}
r & =0.02 \mathrm{~m} \\
r^{2} & =x^{2}+z^{2} \\
r^{2} & =2 x^{2} \\
x & =z=\frac{r}{\sqrt{2}} \\
\frac{V}{g} & =x z=\frac{r}{\sqrt{2}} \frac{r}{\sqrt{2}}=\frac{r^{2}}{2}
\end{aligned}
$$

We are interested in the x coordinate of a point that is on the pole for which $\mathrm{z}(\mathrm{x})=0.5 \mathrm{~mm}$.

$$
\begin{gathered}
\frac{r^{2}}{2}=x z \\
x=\frac{r^{2}}{2 z}=0.4 \mathrm{~m}
\end{gathered}
$$

Note: The field strength is not needed to solve the problem.

## Problem 7

The MAX-II ring has $n=20$ dipoles of $l=1 \mathrm{~m}$ length. Electrons circulating in the ring have $E_{0}=1.5 \mathrm{GeV}$ energy, and the circumference of the ring is $C=90 \mathrm{~m}$. The RF-cavity has a frequency of $f_{R F}=100 \mathrm{MHz}$. Take the average current in the ring to be $I=200 \mathrm{~mA}$.
a) What is the maximum number of bunches can circulate in MAX II?
b) How much charge is there in one bunch and how much in total?
c) How much energy does one electron lose per turn if there are no insertion devices (IDs)? d) What is the power the cavity needs to supply? (electron charge is $e=-1.6 \times 10^{-19} \mathrm{C}$ )

## Solution

a) In the time single bunch completes one turn the cavity completes several cycles in which it can accelerate. Then, the number of bunches can be calculated as RF frequency divided by revolution frequency:

$$
N=\frac{f_{R F}}{f_{\text {rev }}}=\frac{f_{R F} C}{c}=30
$$

b) Current is defined as

$$
I=\frac{\Delta Q}{\Delta t}
$$

To find a charge in a one bunch, following the logic in $a$ ), the time has to be equal to one RF period $\Delta t=1 / f_{R F}$ as it is the time of passage of one bunch if all the ring is filled with the maximum possible number of bunches. Then

$$
\begin{aligned}
Q_{1}=I \Delta t & =I / f_{R F}=2 \mathrm{nC} \\
Q_{\text {total }} & =N Q_{1}=60 \mathrm{nC}
\end{aligned}
$$

c) The energy loss per turn by one electron (for the case of constant bending radius $r$ in bending magnets) is

$$
U_{\text {loss }}^{e^{-}}[\mathrm{MeV}]=8.85 \times 10^{-2} \frac{\left(E_{0}[\mathrm{MeV}]\right)^{4}}{r[\mathrm{~m}]}
$$

Taking $r \approx C / 2 \pi$, we find $U_{\text {loss }}^{e^{-}}=31.3 \mathrm{keV}$.
d) The cavity needs to supply the lost energy $U_{\text {loss }}^{e^{-}}$to each electron in the time of full turn

$$
P=N_{e^{-}} U_{\text {loss }}^{e^{-}} \frac{c}{C}=\frac{Q_{\text {total }}}{e} U_{\text {loss }}^{e^{-}} \frac{c}{C}=6.26 \mathrm{~kW}
$$

