Preparation for summer school

The course Fundamentals of Accelerator Technology gives an introduction to the technology of modern accelerators. This text is a preparation for the course. Go back to your physics books if you need a more detailed description of a section.

The particles in accelerators travel with speeds close to the speed of light and one needs the theory of special relativity to describe their motion. It is not required that you know special relativity when you come to the course. We will explain the parts that you need, but you can prepare yourself by reading the short section below.

The section on magnetic multipoles is new for most of you. Just read this part and try to understand the concepts of dipole, quadrupole and sextupole.

The Lorentz force

In accelerators the charged particles are accelerated by the Lorentz force. This force is the sum of the electric and magnetic force on a charge q that travels with velocity $\boldsymbol{v}$ in a region with electric field $\boldsymbol{E}$ and magnetic flux density $\boldsymbol{B}$

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{1}$$

Since the magnetic force, $q\boldsymbol{v} \times \boldsymbol{B}$, is perpendicular to $\boldsymbol{v}$, it can only change the direction of $\boldsymbol{v}$, but not the speed $v = |\boldsymbol{v}|$.

Quiz: In a region the magnetic flux density is $\boldsymbol{B} = B_0(0.8, 0.6, 0)$, where $B_0 = 1.20$ T, and the electric field is $\boldsymbol{E} = E_0(0, 0, 1)$, where $E_0 = 2.00$ MV/m. Determine the force $\boldsymbol{F}$ on a proton that travels with velocity $\boldsymbol{v} = v_0(0, 0, 1)$ through the region, where $v_0 = 0.70c$. Here c is the speed of light in vacuum.

Answer: $\boldsymbol{F} = qB_0v_0(-0.6, 0.8, 0) + qE_0(0, 0, 1)$ where $qv_0B_0 = 4.05 \cdot 10^{-11}$ N and $qE_0 = 3.22 \cdot 10^{-13}$ N.

The Maxwell equations

The time domain Maxwell equations are

$$\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \times \boldsymbol{H} &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \\
\nabla \cdot \boldsymbol{D} &= \rho \\
\nabla \cdot \boldsymbol{B} &= 0,
\end{aligned} \tag{2}$$

where $\boldsymbol{J}$ is the current density of free charges, ρ is the free space charge density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{D}$ is the electric flux density, $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{B}$ is the magnetic flux density. In order to solve the equations one needs constitutive relations between $\boldsymbol{D}$ and $\boldsymbol{E}$ and between $\boldsymbol{B}$ and $\boldsymbol{H}$. In vacuum these relations are

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \boldsymbol{E} \\ \boldsymbol{B} &= \mu_0 \boldsymbol{H}, \end{aligned} \tag{3}$$

where ε_0 is the permittivity of vaccum and μ_0 the permeablity of vacuum.

RF-waves

Radio frequency waves (RF waves) are time harmonic electromagnetic waves with frequencíes in the range 30 kHz-300 GHz. In accelerators it is common to use RF waves in the frequency range 100 MHz-10 GHz. Another common name for these waves is microwaves. Since RF waves oscillate with a fixed frequency it is convenient to represent them with complex fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, current density $\boldsymbol{J}(\boldsymbol{r})$ and charge density $\rho(\boldsymbol{r})$. The complex fields are related to the real time-dependent physical fields via the transformation $\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}\}$. In vacuum the Maxwell equations for the complex fields are

$$\begin{aligned} \nabla \times \boldsymbol{E}(\boldsymbol{r}) &= \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{r}) \\ \nabla \times \boldsymbol{B}(\boldsymbol{r}) &= \mu_0 \boldsymbol{J}(\boldsymbol{r}) - \mathrm{i}\frac{\omega}{c^2}\boldsymbol{E}(\boldsymbol{r}) \\ \nabla \cdot \boldsymbol{E}(\boldsymbol{r}) &= \frac{1}{\varepsilon_0}\rho(\boldsymbol{r}) \\ \nabla \cdot \boldsymbol{B}(\boldsymbol{r}) &= 0, \end{aligned} \tag{4}$$

where c is the speed of light in vacuum (299 792 458 m/s$\approx 3 \cdot 10^8$ m/s). In addition to these equations one needs boundary conditions to solve the equations. The waves described in the course are traveling inside pipes and cavities with metallic walls. A good enough approximation is that these walls are perfectly conducting and on them the tangential component of the electric field is zero. For a metal surface S with unit normal $\boldsymbol{n}$, the boundary condition then reads

$$\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}, \qquad \boldsymbol{r} \in S. \tag{5}$$

Example:

A plane wave that is linearly polarized in the $x-$direction and propagates in the positive $z-$direction in a region with vacuum has the complex electric field

$$\boldsymbol{E} = E_0 e^{\mathrm{i}kz}\hat{\boldsymbol{x}}.$$

The magnetic field is given by the plane wave rule $\boldsymbol{H} = \eta_0^{-1}\hat{\boldsymbol{z}} \times \boldsymbol{E}$, where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the free space wave impedance. Thus

$$\boldsymbol{H} = \eta_0^{-1} E_0 e^{\mathrm{i}kz}\hat{\boldsymbol{y}}.$$

Applying the rule $\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}\}$, the time domain fields are

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) &= E_0\cos(\omega t - kz)\hat{\boldsymbol{x}} \\ \boldsymbol{H}(\boldsymbol{r},t) &= \eta_0^{-1}E_0\cos(\omega t - kz)\hat{\boldsymbol{y}}. \end{aligned}$$

Electrostatics

In the static case all time derivatives in the Maxwell equations are zero and the electric and magnetic fields decouple. We use the same notation for the static fields as for the complex fields, $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$. Often there is no risk for confusion. The static electric field in vacuum is given by the two equations

$$\begin{aligned} \nabla \times \boldsymbol{E} &= \boldsymbol{0} \\ \nabla \cdot \boldsymbol{E} &= \frac{1}{\varepsilon_0}\rho. \end{aligned} \tag{6}$$

We can introduce the scalar electric potential Φ_{E} as $\boldsymbol{E} = -\nabla\Phi_{\mathrm{E}}$, where Φ_{E} satisfies Poissons equation

$$\nabla^2\Phi_{\mathrm{E}} = -\frac{1}{\varepsilon_0}\rho. \tag{7}$$

In accelerator technology the electrostatic fields are less important than the magnetostatic fields.

Magnetostatics

The static magnetic field $\boldsymbol{H}$ and magnetic flux density $\boldsymbol{B}$ satisfy

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \boldsymbol{J} \\ \nabla \cdot \boldsymbol{B} &= 0. \end{aligned} \tag{8}$$

In a source free region, where $\boldsymbol{J} = \boldsymbol{0}$, we can introduce the scalar magnetic potential Φ_{M} as $\boldsymbol{B} = -\nabla\Phi_{\mathrm{M}}$ and solve Laplace equation $\nabla^2\Phi_{\mathrm{M}} = 0$. The second alternative, which is more common and general, is to introduce the magnetic vector potential $\boldsymbol{A}$ as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. This can be used also in regions where $\boldsymbol{J} \neq \boldsymbol{0}$. Then $\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0\boldsymbol{J}$. By enforcing $\nabla \cdot \boldsymbol{A} = 0$ and using $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$, the equation for $\boldsymbol{A}$ follows

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J}. \tag{9}$$

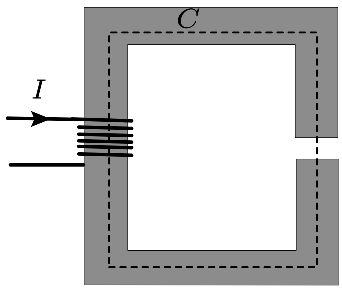
There is also a third alternative to find $\boldsymbol{H}$, and that is to use Ampère's law in integral form

$$\oint_{\mathcal{C}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell} = \int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}S, \tag{10}$$

where $\mathcal{C}$ is a closed curve that spans the surface S. The direction of the unit normal vector $\hat{\boldsymbol{n}}$ to S is related to the direction of the curve $\mathcal{C}$ via the right hand rule.

Magnetic circuits

The particles are guided along prescribed trajectories by static magnetic fields. These fields are either generated by permanent magnets or by magnetic circuits that consists of iron cores and coils. We now describe how one can produce a magnetic dipole field, which is a constant magnetic field. It is used for bending the beam of particles.



Consider the circuit in the figure. The grey part is iron and the white is air, or vacuum. The air gap d is small. We assume that the permeability of the iron is very large and that the leakage of magnetic flux is negligible. Ampère's law (10) gives

$$NI = \oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{11}$$

where C is the closed curve in the figure, N the number of windings in the coil, and I the current. The normal component of the magnetic flux density $\boldsymbol{B}$ is always continuous over a surface. Thus it is approximately the same magnetic flux inside the iron and in the air gap. Since $\boldsymbol{H} = \dfrac{1}{\mu_0\mu_r}\boldsymbol{B}$ in the iron and $\boldsymbol{H} = \dfrac{1}{\mu_0}\boldsymbol{B}$ in air, it means that the magnetic field is approximately μ_r times stronger in the air gap than in the iron. The integral over the part of C that is in the iron is neglected since $\mu_r \gg 1$. The magnetic flux density is approximately constant in the air gap and directed along C. From this we get the magnetic flux density in the air gap

$$\boldsymbol{B} = \frac{NI}{d\mu_0}\hat{\boldsymbol{y}}, \tag{12}$$

where the unit vector $\hat{\boldsymbol{y}}$ is is directed downwards.

Special theory of relativity

According to the special theory of relativity, a number of physical quantities have different values when they are measured in two systems that move with constant velocity relative each other. Time, length, impulse, the electric field, the magnetic field, and charge density are examples of such quantities. This is a consequence of that the speed of light has the same value for all systems.

Consider a system S and a system S'. An observer in system S sees system S' move with constant velocity $\boldsymbol{v}$. The observer in system S' has a rod of length ℓ_0 that is directed parallell with $\boldsymbol{v}$ and he also brings a clock. The observer in system S sees the rod pass by and measures its length to ℓ. He also measures the time T it takes for the clock in S' to run a time T_0. He finds

$$\begin{aligned} \ell &= \frac{\ell_0}{\gamma} \\ T &= \gamma T_0, \end{aligned} \tag{13}$$

where

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}. \tag{14}$$

It means that seen from S the rod becomes shorter by a factor of γ^{-1} and the clock in S' is seen to run slower by the same factor.

If the observer in S' rotates the rod so that it is perpendicular to $\boldsymbol{v}$, then both observers will measure the same length ℓ_0 of the rod. Quantities that have the same value in the two systems are said to be invariant under the transformation between the systems. Charge and the speed of light are invariant. Also the laws of physics, like the Maxwell equations, are invariant.

Transformation of fields

Now assume that the two observers also measure the electric and magnetic fields in their systems. The observer in S measures the electric field $\boldsymbol{E} = \boldsymbol{E}_\perp + \boldsymbol{E}_\parallel$ and the magnetic flux density $\boldsymbol{B} = \boldsymbol{B}_\perp + \boldsymbol{B}_\parallel$, where $\boldsymbol{E}_\perp$ is the electric field perpendicular to $\boldsymbol{v}$ and $\boldsymbol{E}_\parallel$ is the electric field parallell to $\boldsymbol{v}$. Then the fields measured by the observer in system S' are

$$\begin{cases} \boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v} \times \boldsymbol{B}) \\ \boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \\ \boldsymbol{B}'_\perp = \gamma(\boldsymbol{B}_\perp - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}) \\ \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel. \end{cases} \tag{15}$$

It means that the electric and magnetic field components parallell to the velocity $\boldsymbol{v}$ are invariant whereas the components perpendicular to $\boldsymbol{v}$ are not. Equation (15) is the transformation of the electromagnetic fields from the system S to the system S'.

Quiz: Obviously the observer in S' sees system S traveling with velocity $-\boldsymbol{v}$ relative S'. It must then be that the transformation from S' to S is:

$$\begin{cases} \boldsymbol{E}_\perp = \gamma(\boldsymbol{E}'_\perp - \boldsymbol{v} \times \boldsymbol{B}') \\ \boldsymbol{E}_\parallel = \boldsymbol{E}'_\parallel \\ \boldsymbol{B}_\perp = \gamma(\boldsymbol{B}'_\perp + \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}') \\ \boldsymbol{B}_\parallel = \boldsymbol{B}'_\parallel. \end{cases} \tag{16}$$

Consider the case when $\boldsymbol{B} = B_0\hat{\boldsymbol{x}}$, $\boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{v} = v\hat{\boldsymbol{z}}$. Insert this into (15) to get $\boldsymbol{B}'$ and $\boldsymbol{E}'$. Then insert $\boldsymbol{B}'$ and $\boldsymbol{E}'$ into (16) and check that you get back $\boldsymbol{B} = B_0\hat{\boldsymbol{x}}$ and $\boldsymbol{E} = \boldsymbol{0}$.

Quiz: The length of the linear accelerator (LINAC) in MAXIV is $L = 340$ m. When the electrons exit the LINAC and enter the storage ring, they move with the speed $v = 0.999999985c$.

a) What is the length L' of the LINAC when measured by an observer in a system S' traveling with the speed $v = 0.999999985c$ along the LINAC?

b) The observer brings a clock and measures the time T_0 it takes for him to travel along the LINAC. What is this time?

c) An observer that is at a fixed position relative the LINAC measures the time T it takes for S' to travel along the LINAC. What is this time? Check that it is related to T_0 as in (13).

Answer: a) $L' = 5.9$ cm.

b) $T_0 = L'/v = 1.96 \cdot 10^{-10}$ s

c) $T = L/v = 1.13 \cdot 10^{-6}$ s. This is the same as γT_0

Equation of motion

A particle with rest mass m_0 that moves with velocity $\boldsymbol{v}$ relative an observer has the impulse

$$\boldsymbol{p} = \gamma m_0 \boldsymbol{v}. \tag{17}$$

When the force on the particle is $\boldsymbol{F}$, the equation of motion is

$$\frac{d\boldsymbol{p}(t)}{dt} = \boldsymbol{F}. \tag{18}$$

Notice that γ depends on time since $\boldsymbol{v}$ is time dependent.

Circular motion of a charge in a constant magnetic field

Assume a region with constant magnetic flux density $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ and no electric field. A particle with charge q and mass m_0 travels with velocity $\boldsymbol{v} = v_x(t)\hat{\boldsymbol{x}} + v_y(t)\hat{\boldsymbol{y}}$ in the region. The speed of the particle is constant since the electric field is zero. The magnetic field is perpendicular to the velocity and hence $q|\boldsymbol{v} \times \boldsymbol{B}|$ is constant. The particle then moves in a circle. We can find the radius R of the circle by solving the equation of motion. This is the same solution that is given in the basic courses in mechanics. The only difference is that the mass m_0 is exchanged for γm_0. A simpler way is to say that the centrifugal force $\gamma m_0 v^2/R$ is equal to the magnetic force qvB_0, where $v = |\boldsymbol{v}|$. Then the radius is

$$R = \frac{\gamma m_0 v}{qB_0}. \tag{19}$$

It takes a time $T = 2\pi R/v$ for the particle to travel one lap in the circle. The angular frequency for the particle is $\omega = \dfrac{2\pi}{T}$ and then

$$\omega = \frac{qB_0}{\gamma m_0}. \tag{20}$$

It is somewhat more complicated to solve the equation of motion. We need initial conditions in order to get the unique solution for the path of the particle. We let these conditions be that the particle passes the origin with speed $\boldsymbol{v}(0) = v(1, 0, 0)$ at time $t = 0$.

The magnetic force is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = q(v_y(t)\hat{\boldsymbol{x}} - v_x(t)\hat{\boldsymbol{y}})B_0. \tag{21}$$

Since v is constant, the equation of motion gives

$$\gamma m_0 \left(\frac{dv_x(t)}{dt}\hat{\boldsymbol{x}} + \frac{dv_y(t)}{dt}\hat{\boldsymbol{y}}\right) = q(v_y(t)\hat{\boldsymbol{x}} - v_x(t)\hat{\boldsymbol{y}})B_0. \tag{22}$$

From this we get two coupled equations

$$\begin{aligned} \gamma m_0 \frac{dv_x(t)}{dt} &= qv_y(t)B_0 \\ \gamma m_0 \frac{dv_y yt)}{dt} &= -qv_x(t)B_0. \end{aligned} \tag{23}$$

Elimination of v_y gives the equation for v_x and elimination of v_x the equation for v_y:

$$\begin{aligned} \frac{d^2 v_x(t)}{dt^2} + \left(\frac{qB_0}{\gamma m_0}\right)^2 v_x(t) = 0 \\ \frac{d^2 v_y(t)}{dt^2} + \left(\frac{qB_0}{\gamma m_0}\right)^2 v_y(t) = 0. \end{aligned} \tag{24}$$

The solution that satisfies the initial condition $\boldsymbol{v}(0) = v(1,0,0)$ is

$$\boldsymbol{v}(t) = v(\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}), \tag{25}$$

where $\omega = \dfrac{qB_0}{\gamma m_0}$. The equation of the circle is obtained from an integration of (25) and the initial condition $\boldsymbol{r}(0) = (0,0,0)$:

$$\boldsymbol{r}(t) = \frac{v}{\omega}\left(\sin(\omega t)\hat{\boldsymbol{x}} - (\cos(\omega t) - 1)\hat{\boldsymbol{y}}\right). \tag{26}$$

The radius of the circle is $R = \dfrac{v}{\omega} = \dfrac{\gamma m_0 v}{qB_0}$, as we already seen in (19).

Quiz Assume an electron traveling with speed $v = 0.999999985c$ in a region with constant magnetic flux density $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$. The velocity of the electron is perpendicular to $\boldsymbol{B}$. Determine B_0 if the radius of the electron's orbit is 84 m.

Answer: $B_0 = 0.117$ T. The speed and the radius are the ones used in the MAX IV storage ring. In reality the orbit is not circular but consists of 20 bends and 20 straight sections. It means that the magnetic flux density in the bends is stronger than B_0.

Electron volt

The kinetic energies of charged particles are measured in electron volt (eV). One electron volt is the energy an elementary charge gains when it is accelerated by a voltage of one volt. By that 1eV=$1.609 \cdot 10^{-19}$ J. According to special relativity the total energy of a mass m_0 traveling with speed v is $W = \gamma m_0 c^2$. The kinetic energy is the total energy minus the energy of the mass when it is at rest

$$W_{\text{kin}} = \gamma m_0 c^2 - m_0 c^2. \tag{27}$$

Quiz: The electrons in the MAX IV storage ring have an energy of 3 GeV. What is $\beta = v/c$, where v is the speed of the electrons?

Answer: 0.999999985

Quiz: The protons in the ESS accelerator will have an energy of 3 GeV when they hit the target. What is $\beta = v/c$, where v is the speed of the protons?

Answer: 0.95

Multipole expansions of the magnetic field

Along an accelerator there are several regions with magnetic fields where the beam of particles is focused or bent. These regions can often be considered to be two-dimensional, since the magnetic field is almost constant along the beam in each region. Assume a two-dimensional region with vacuum around the beam. Since there are no currents, Ampère's law says that $\nabla \times \boldsymbol{B} = 0$. In that region we may express $\boldsymbol{B}$ in terms of the scalar magnetic potential Φ_M

$$\boldsymbol{B}(x, y) = -\nabla \Phi_\mathrm{M}(x, y). \tag{28}$$

Since $\nabla \cdot \boldsymbol{B} = 0$ it follows that the scalar magnetic potential satisfies the Laplace equation

$$\nabla^2 \Phi_\mathrm{M}(x, y) = 0. \tag{29}$$

We now use polar coordinates ρ, φ where $\rho = \sqrt{x^2 + y^2}$ and φ is the azimuthal angle measured from the $x-$axis. The general solution to Laplace equation is obtained by the method of separation of variables. This standard method gives

$$\Phi_\mathrm{M}(\rho, \varphi) = \alpha_0 + \sum_{n=1}^{\infty} \rho^n (\alpha_n \cos n\varphi + \beta_n \sin n\varphi), \tag{30}$$

where α_n and β_n are constants.

The two terms proportional to ρ are dipole terms, the two proportional to ρ^2 are quadrupoles, the two proportional to ρ^3 are sextupoles, and so on. In accelerators the dipole fields are used for bending the beam of particles, quadrupoles to focus the beam, and sextupoles to further correct the cross section of the beam.

The magnetic flux density is given by $\boldsymbol{B} = -\nabla \Phi_\mathrm{M}$, which in polar coordinates reads

$$\boldsymbol{B}(\rho, \varphi) = -\frac{\partial \Phi_\mathrm{M}(\rho, \varphi)}{\partial \rho} \hat{\boldsymbol{\rho}} - \frac{1}{\rho} \frac{\partial \Phi_\mathrm{M}(\rho, \varphi)}{\partial \varphi} \hat{\boldsymbol{\varphi}}. \tag{31}$$

This gives

$$\boldsymbol{B}(\rho, \phi) = -n\rho^{n-1} \left((\alpha_n \cos n\varphi + \beta_n \sin n\varphi) \hat{\boldsymbol{\rho}} - (\alpha_n \sin n\varphi - \beta_n \cos n\varphi) \hat{\boldsymbol{\varphi}} \right). \tag{32}$$

The magnetic field of a dipole ($n = 1$) is a constant vector, the magnetic field of a quadrupole ($n = 2$) is proportional to ρ, the magnetic field of a sextupole ($n = 3$) is proportional to ρ^2, and so on.

Example:

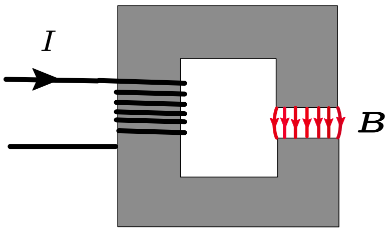
We can express the lowest order multipoles in terms of the cartesian coordinates. For the dipole, $n = 1$, we get

$$\begin{aligned} \rho \cos \varphi &= x \\ \rho \sin \varphi &= y. \end{aligned} \tag{33}$$

and from that and (30) the dipole magnetic flux density is

$$\boldsymbol{B}_1(x,y) = -\nabla\Phi_{\mathrm{M}} = -\alpha_1\hat{\boldsymbol{x}} - \beta_1\hat{\boldsymbol{y}}. \tag{34}$$

This field is a constant vector in space. In the figure below a magnetic circuit with two magnetic poles, that generates the dipole field for $\alpha_1 = 0$ and $\beta_1 \neq 0$ is shown.



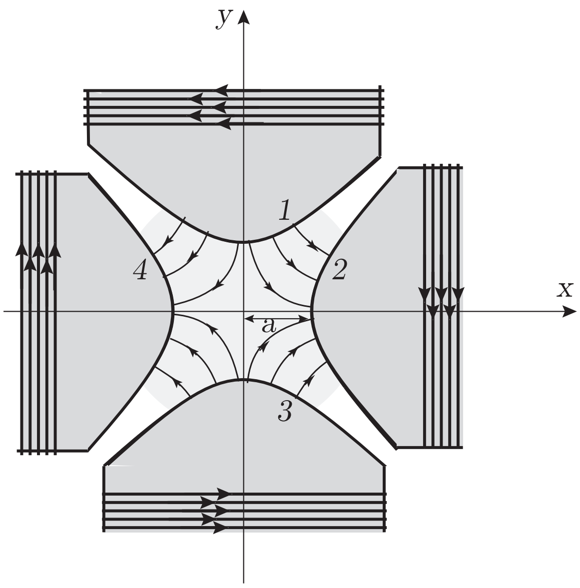
For the quadrupole, $n = 2$, we have

$$\begin{aligned} \rho^2\cos 2\varphi &= \rho^2(\cos^2\varphi - \sin^2\varphi) = x^2 - y^2 \\ \rho^2\sin 2\varphi &= 2\rho^2\cos\varphi\sin\varphi = 2xy. \end{aligned} \tag{35}$$

The magnetic flux density is

$$\boldsymbol{B}_2(x,y) = 2\alpha_2(y\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}) - 2\beta_2(y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}). \tag{36}$$

In the figure below the quadrupole for $\alpha_2 \neq 0$ and $\beta_2 = 0$ is shown. It is generated by four iron cores with coils (four magnetic poles). On the curved surface of an iron core the potential is constant. That gives the shape of the surface as $\rho = \mathrm{const}/\cos 2\phi$ on two opposite surfaces and $\rho = -\mathrm{const}/\cos 2\phi$ on the other two. The case $\alpha_2 = 0$ and $\beta_2 \neq 0$ gives the same field but rotated 45°.



For the sextupole, $n = 3$, we get

$$\begin{aligned}
\rho^3 \cos 3\varphi &= \rho^3(\cos 2\varphi \cos\varphi - \sin 2\varphi \sin\varphi) = x^3 - 3xy^2 \\
\rho^3 \sin 3\varphi &= \rho^3(\sin 2\varphi \cos\varphi + \cos 2\varphi \sin\varphi) = 3x^2y - y^3.
\end{aligned} \tag{37}$$

The magnetic flux density is

$$\boldsymbol{B}_3(x, y) = -3\alpha_3 \left((x^2 - y^2)\hat{\boldsymbol{x}} - 2xy\hat{\boldsymbol{y}}\right) - 3\beta_3 \left(2xy\hat{\boldsymbol{x}} + (x^2 - y^2)\hat{\boldsymbol{y}}\right). \tag{38}$$

The sextupole field can be generated by six magnetic poles, in the same manner as the quadrupole.