Nordic Particle Accelerator School

Exercises

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$1 \mathbf{RF}$

1.1

A circular cylindric cavity has radius a and length h. The electric field is from the TM_{010} mode, and is given by

$$\boldsymbol{E}(\rho,t) = E_0 J_0(k\rho) \cos(\omega t + \phi) \hat{\boldsymbol{z}},$$

where ϕ is the phase angle, $(-180 \leq \phi < 180^{\circ})$ and k = 2.405/a. A particle with speed v enters the cavity at t = 0 and travels through the cavity, along the symmetry axis. The acceleration is small enough so that the speed of the proton can be considered to be constant.

- a) Determine the energy gain of a proton when it has travelled through the cavity. Express the energy in terms of E_0 , v, a, h, and ϕ .
- b) For certain h the energy gain is zero, regardless of the phase angle ϕ . Determine these h.
- c) Assume that $h \ll a$ and $v \sim c$. What value of ϕ maximises the energy gain for the proton?
- d) Assume that $h \ll a$, $v \sim c$, and that electrons are accelerated in the cavity. What value of ϕ maximises the energy gain for the electron?



e) In linear accelerators the particles come in bunches. The bunch gets stretched out when it travels through a pipe between cavities, since the distance between the faster and slower particles increases with time. Assume that $h \ll a$ and $v \sim c$. A short bunch enters the cavity in a small time interval around t = 0. The curves in the figure are for six different phase angles ϕ . Curve 2 corresponds to the ϕ obtained in c). Which of the three curves 1, 2 and 3 do you think is suitable for accelerating the bunch.

- f) Which of the three curves 4,5 and 6 is suitable for accelerating a bunch of electrons in a linear accelerator? Curve 5 corresponds to the ϕ in d).
- **g)** For storage rings, like the one at MAX IV, one might believe that one should choose the same curve as in f). That is not the case. Which curve is suitable and why?

1.2

The TE₁₀ mode in a rectangular waveguide with cross section 0 < x < a, 0 < y < b (a > b), filled with air, has the complex electric field

$$oldsymbol{E}(oldsymbol{r}) = E_0 \sin\left(rac{\pi x}{a}
ight) e^{\mathrm{i}k_z z} \hat{oldsymbol{y}}$$

where $k_z = \sqrt{k^2 - (\pi/a)^2}$ is the z-component of the wave vector and $k = \omega/c$ is the wavenumber.

- a) What is the time-domain electric field $E(\mathbf{r},t)$? (Time dependence is $e^{-i\omega t}$)
- b) Below a certain frequency (the cut-off frequency) the wave will attenuate with increasing z. What is this frequency if a = 10 cm?
- c) What is the phase speed in the z-direction of the TE_{10} mode when a = 10 cm and f = 2 GHz?
- d) Is the phase speed larger or smaller than the speed of light? Is there a contradiction with the theory of special relativity?
- e) Assume that you like to feed the TE_{10} in the waveguide by a coaxial cable. You drill a small hole in the waveguide and attach the coaxial cable. The inner conductor of the coaxial cable extends straight into the waveguide. Suggest a suitable position (x, y) for the hole.

1.3

The boundary condition for electromagnetic waves at a perfectly conducting surface is that the tangential component of the electric field is zero. From this, and the Maxwell equations, one can derive the following boundary conditions at the surfaces of a rectangular waveguide:

• The normal component of the magnetic field is zero on the surfaces $(\hat{\boldsymbol{n}} \cdot \boldsymbol{H} = 0)$.

- The normal derivative of the normal component of the electric field is zero on the surfaces $(\hat{\boldsymbol{n}} \cdot \nabla(\hat{\boldsymbol{n}} \cdot \boldsymbol{E} = 0))$.
- The normal derivative of the tangential components of the magnetic field are zero on the surfaces $(\hat{\boldsymbol{n}} \cdot \nabla(\hat{\boldsymbol{n}} \times \boldsymbol{H}) = \boldsymbol{0}).$

This imply that all components of the electric and magnetic fields have the x-dependence

$$\sin\left(\frac{m\pi x}{a}\right)$$
 or $\cos\left(\frac{m\pi x}{a}\right)$

and y-dependence

$$\sin\left(\frac{n\pi y}{b}\right)$$
 or $\cos\left(\frac{n\pi y}{b}\right)$.

The z-dependence is $e^{ik_z z}$ for all componants.

a) The TE_{mn} modes have $E_z = 0$ whereas H_z has the space dependence

$$\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}$$

Determine the space dependences of E_x , E_y , H_x , and H_y .

- **b)** The TM_{mn} modes have $H_z = 0$ whereas $E_z \neq 0$. Determine the space dependences of E_x , E_y , E_z , H_x , and H_y .
- c) Can there be TM_{m0} and TM_{0n} modes in a rectangular waveguide?

1.4

A rectangular parallelpiped cavity given by 0 < x < a, 0 < y < b, 0 < z < h can be viewed as a rectangular waveguide with metallic walls at z = 0 and z = h. The same conditions as given in the previous problem are valid for all six walls. The TM-modes have $H_z = 0$ whereas E_z has the space dependence $\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{\ell\pi z}{h}\right)$ where m, n, ℓ can take the values $m = 1, 2..., n = 1, 2, ..., \ell = 0, 1, 2...$

- a) Determine the space dependence of E_x , E_y , H_x , and H_y for the TM-modes.
- **b)** Determine the resonance frequency for the $\text{TM}_{mn\ell}$ mode, expressed in a, b, h, m, n, ℓ , and the speed of light c.
- c) The TM₀₁₀ mode of a circular cylindrical cavity has $E_x = 0$, $E_y = 0$, and $E_z \neq 0$. Is there any mode for the the parallelpiped that has similar fields? Would it be possible to use this cavity and mode in accelerators?

1.5

The group speed of a waveguide is the speed the energy travels with in a waveguide. It is given by

$$v_g = \left(\frac{dk_z}{d\omega}\right)^{-1}$$

Show that $v_g = \frac{k_z}{k}c$.

Hint: Remember that k_t is a constant and that $k^2 = k_t^2 + k_z^2$. Try also to find a simple explanation to why $v_q/c = k_z/k$.

1.6

A coaxial waveguide has an inner conductor with radius a and an outer conductor with inner radius b. There is vacuum between the conductors. The electric field of the TEM-mode traveling in the waveguide is

$$\boldsymbol{E}(\rho, z, t) = E_0 \frac{a}{\rho} \cos(\omega t - kz) \hat{\boldsymbol{\rho}}$$

- a) Determine the magnetic field $H(\rho, z, t)$ of the TEM mode. Use (6) belw.
- **b**) Show that the relation between the electric and magnetic fields of the TEM mode is the same as for a plane wave

$$\boldsymbol{H}(\boldsymbol{r},t) = \eta_0^{-1} \hat{\boldsymbol{z}} \times \boldsymbol{E}(\boldsymbol{r},t)$$

- c) What is the voltage v(z,t) between the inner and outer conductor? (*Hint:* Integrate E from the inner to the outer surface in the radial direction and use $E(\mathbf{r},t) = -\nabla v(\mathbf{r},t)$.)
- **d)** What is the current i(z,t) along the inner conductor? (*Hint:* Use the Ampere circuit law $\oint_C \boldsymbol{H} \cdot d\boldsymbol{\ell} = i$.)
- e) The ratio v(z,t)/i(z,t) for a wave in a transmission line is called the characteristic impedance of the line. Determine Z₀ for the coaxial waveguide.
 Comment The characteristic impedance is important in microwave technique. Maybe you have heard of 50 Ω cable. It means that the characteristic impedance is 50 Ω. When a 50 Ω cable is hooked up to a device with input impedance 50 Ω then there will be no reflected wave.

1.7

The attenuation of waves is often measured in decibel (dB). The attenuation, measured in dB, of a wave traveling from z = 0 to $z = z_1$ is

$$20\log\left(\frac{|\boldsymbol{E}(x,y,z_1)|}{|\boldsymbol{E}(x,y,0)|}\right),\,$$

where $E(\mathbf{r})$ is the complex electric field and log is the logarithm with base 10.

At normal temperatures the walls of waveguides are not perfectly conducting. A wave propagating through the waveguide looses energy since the surface currents heat up the walls of the waveguide. The wave is by that attenuated as

$$\boldsymbol{E}(x,y,z) = \boldsymbol{E}(x,y,0)e^{-\alpha z}e^{\mathrm{i}\beta z}$$

where α is the attenuation constant and β the phase constant.

- a) Assume that $\alpha = 10^{-2} \text{ m}^{-1}$ for a certain waveguide. How many dB has the wave been damped when it has travelled a distance of 20 m in the waveguide?
- b) A waveguide consists of two parts, the first with length L_1 and the second with length L_2 . The attenuation in the first part is 0.03 dB/m and in the second 0.04 dB/m. How many dB has a wave been attenuated when it has travelled through both parts if $L_1 = 10$ m and $L_2 = 5$ m?

(*Hint:* You can solve this in 5 seconds.)

1.8

Show that the time average of the electric and magnetic energy for the TM_{010} mode in a cylindric cavity are equal.

Hint: Use some of the formulas below.

Formulas for RF

The time domain Maxwell equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0,$$
(1)

The frequency domain Maxwell equations

Using the time dependence $e^{-i\omega t}$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = i\omega \boldsymbol{B}(\boldsymbol{r})$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r}) - i\omega \boldsymbol{D}(\boldsymbol{r})$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\varepsilon_0} \rho(\boldsymbol{r})$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0,$$

(2)

The constitutive relations for vacuum

$$D = \varepsilon_0 E$$

$$B = \mu_0 H$$
(3)

The wave impedance of vacuum

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{4}$$

The plane wave relation in vacuum

$$\boldsymbol{H} = \eta_0^{-1} \hat{\boldsymbol{k}} \times \boldsymbol{E} \tag{5}$$

for a plane wave $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. Here $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ is the unit vector in the direction of propagation.

Cylindric coordinates (ρ, ϕ, z)

Curl of
$$\boldsymbol{A}(\rho,\phi,z) = \hat{\boldsymbol{\rho}}A_{\rho}(\rho,\phi,z) + \hat{\boldsymbol{\phi}}A_{\phi}(\rho,\phi,z) + \hat{\boldsymbol{z}}A_{z}(\rho,\phi,z)$$
 is

$$\nabla \times \boldsymbol{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_{z}}{\partial\phi} - \frac{\partial A_{\phi}}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_{z}}{\partial\rho}\right) + \hat{\boldsymbol{z}}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\rho A_{\phi}\right) - \frac{\partial A_{\rho}}{\partial\phi}\right)$$
(6)

 TM_{010} mode of a cylindric cavity

where k = 2.405/a, a =radius of cylinder.

Hollow waveguides

- \boldsymbol{E} and \boldsymbol{H} are $\sim e^{\mathrm{i}k_z z}$.
- Wave vector $\boldsymbol{k} = \boldsymbol{k}_t + k_z \hat{\boldsymbol{z}}$.
- $k = \sqrt{k_t^2 + k_z^2}$
- Cut of frequency when $k_z = 0 \Rightarrow f_c = \frac{c}{2\pi}k_t$
- TE-modes have $E_z = 0$. TM modes have $H_z = 0$
- Phase speed $v_{\rm p} = \frac{\omega}{k_z}$.

Stored energy

The time average of the stored electric energy, $W_{\rm E}$ and the stored magnetic energy $W_{\rm M}$ in cavity with volume V are

$$W_E = \frac{1}{4} \int_V \boldsymbol{E} \cdot \boldsymbol{D}^* \, \mathrm{d}V$$

$$W_M = \frac{1}{4} \int_V \boldsymbol{B} \cdot \boldsymbol{H}^* \, \mathrm{d}V$$
(8)

Bessel functions

The cylindrical Bessel function of order m satisfies the equation

$$z^{2}\frac{d^{2}}{dz^{2}}J_{m}(z) + z\frac{d}{dz}J_{m}(z) + (z^{2} - m^{2})J_{m}(z) = 0$$
(9)

where m is assumed integer. A useful relation is

$$J_1(z) = -J_0'(z)$$
(10)

RF: answers and solutions

S1.1

a) Since z = vt, $J_0(0) = 1$, $\mathbf{F} = q\mathbf{E}$ and the energy is the line integral of the force, we get the energy gain

$$W = \int_0^h qE(0, z/v) \,\mathrm{d}z = qE_0 \int_0^h \cos(\omega z/v + \phi) \,\mathrm{d}z$$

where $q = 1.609 \cdot 10^{-19}$ As is the elementary charge. The solution is

$$W = qE_0 \frac{v}{\omega} \left(\sin(\omega h/v + \phi) - \sin(\phi) \right)$$

Since k = 2.405/a and $k = \omega/c$ it follows that $\omega = 2.405c/a$ and

$$W = qE_0 \frac{va}{2.405c} \left(\sin\left(\frac{2.405ch}{va} + \phi\right) - \sin(\phi) \right)$$

b) When $\frac{2.405c}{va}h = N2\pi$, where N is a positive integer. Then $h = \frac{va}{2.405c}N2\pi$.

c) The energy has its maximum when $\frac{dW}{d\phi} = 0$. That gives the equation for ϕ

$$\cos\left(\frac{2.405ch}{va} + \phi\right) = \cos(\phi)$$

The solution is $\phi = -\frac{2.405ch}{2va}$.

d) The energy for an electron is

$$W = -qE_0 \frac{va}{2.405c} \left(\sin\left(\frac{2.405ch}{va} + \phi\right) - \sin(\phi) \right)$$

or

$$W = qE_0 \frac{va}{2.405c} \left(\sin\left(\frac{2.405ch}{va} + \phi - \pi\right) - \sin(\phi - \pi) \right)$$

The energy has its maximum when $\frac{dW}{d\phi} = 0$, which gives $\phi = \pi - \frac{2.405ch}{2va}$.

e) Curve 3 is the best. The bunch will be compressed since the slow protons come in later than the fast ones and by that get a stronger electric field.

f) Curve 6 is the best.

g) The fast electrons get a larger radius at all bends than the slow ones, due to the centrifugal force. By that they travel a longer distance and enter the cavity at a later time than the slow electrons. In order to get a compression of the bunch Curve 4 is the best.

S1.2

a) $\boldsymbol{E}(\boldsymbol{r},t) = \operatorname{Re}\{\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}\}$ and then

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0 \sin\left(\frac{\pi x}{a}\right) \cos(\omega t - k_z z) \hat{\boldsymbol{y}}$$

b) Cut-off when $k_z = 0$ gives $k = \pi/a$ and $f_c = c/(2a)$. Since a = 10 cm

$$f_c = 1.5 \,\mathrm{GHz}$$

c) Phase speed= $\frac{\omega}{k_z}$, where $\omega = 4\pi \cdot 10^9$ rad/s and $k_z = \sqrt{k^2 - (\pi/a)^2}$. Thus $v_p = 4.54 \cdot 10^8$ m/s.

d) The phase speed is 50% larger than the speed of light. The special theory of relativity is not violated since the phase speed is not the speed the information and power move with.

e) The charges on the inner conductor couples to the electric field of the TM₁₀ mode. It should then be placed where the electric field is strong. That is either on the lower surface at (x, y) = (a/2, 0) or on the upper surface at (x, y) = (a/2, b).

S1.3

a)

$$E_x \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$E_y \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$H_x \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$H_y \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$

b)

$$E_x \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$E_y \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$E_z \sim \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$H_x \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$
$$H_y \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{ik_z z}$$

S1.4

a)

$$E_x \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{\ell\pi z}{h}\right)$$
$$E_y \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{\ell\pi z}{h}\right)$$
$$H_x \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{\ell\pi z}{h}\right)$$
$$H_y \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{\ell\pi z}{h}\right)$$

b)

$$f_{mn\ell} = \frac{c}{2\pi}\sqrt{(m/a)^2 + (n/b)^2 + (\ell/c)^2}$$

where $m = 0, 1, 2..., m = 0, 1, 2..., \ell = 1, 2...$ and $(m, n) \neq (0, 0)$.

c) The TM₁₁₀ mode has $E_x = 0$, $E_y = 0$ and $E_z \neq 0$. The electric field has its maximal amplitude along the line (a/2, b/2, z) and along this line the magnetic field is zero. This is very similar to the TM₀₁₀ mode of a circular cylinder. One reason why this mode is not used is that the cylinder has less losses than the parallelpiped.

S1.5

For a plane wave the velocity \boldsymbol{v} and the wave vector \boldsymbol{k} are both directed in the direction of propagation. Then $\frac{v_x}{c} = \frac{k_x}{k}$, $\frac{v_y}{c} = \frac{k_y}{k}$ and $\frac{v_z}{c} = \frac{k_z}{k}$. We can see the mode as a superposition of plane waves, each having the same k_z and k. The group speed v_g is the same as v_z and then $v_g = \frac{k_z}{k}c$ for all these waves and for the mode.

S1.6

a) The complex electric field is $\boldsymbol{E}(\boldsymbol{r}) = E_0 \frac{a}{\rho} e^{ikz} \hat{\boldsymbol{\rho}}$ The induction law gives

$$oldsymbol{H} = -\mathrm{i}rac{1}{\omega\mu_0}
abla imes oldsymbol{E}$$

From the given formula for the curl and the relation $k/(\omega\mu_0) = \sqrt{\epsilon_0/\mu_0} \equiv \eta_0^{-1}$

$$\boldsymbol{H} = \eta_0^{-1} E_0 \frac{a}{\rho} e^{\mathrm{i}kz} \hat{\boldsymbol{\phi}}$$

By $\boldsymbol{H}(\boldsymbol{r},t) = \operatorname{Re}\{\boldsymbol{H}(\boldsymbol{r})e^{-\mathrm{i}\omega t}\}$ the time domain magnetic field is

$$\boldsymbol{H}(\boldsymbol{r},t) = \eta_0^{-1} E_0 \frac{a}{\rho} \cos(\omega t - kz) \hat{\boldsymbol{\phi}}$$

- b) This follows from $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}$.
- c) The voltage is

$$v(z,t) = \int_{a}^{b} \hat{\boldsymbol{\rho}} \cdot \boldsymbol{E}(\boldsymbol{r},t) \,\mathrm{d}\boldsymbol{\rho}$$

This gives

$$v(z,t) = E_0 a \ln(b/a) \cos(\omega t - kz)$$

d) The Ampere circuit law and the magnetic field in c) gives

$$i(z,t) = \eta_0^{-1} E_0 2\pi a \cos(\omega t - kz)$$

e) From c) and d) we get $Z_0 = \eta_0 (2\pi)^{-1} \ln(b/a)$.

S1.7

a) $20 \log(\exp(-10^{-2} \cdot 20)) = -1.74 \text{ dB}$ (the wave is damped 1.74 dB)

b) We get $10\cdot 0.03 + 5\cdot 0.04 = 0.5$ dB. This is the big advantage with the dB scale. You just add.

2 Basic physics

$\mathbf{2.1}$

The electrons at MAX IV reach a kinetic energy of 3 GeV, the protons at ESS reach 2.5 GeV and the protons at LHC reach 6.5 TeV. What is $\beta = v/c$ for these particles?

$\mathbf{2.2}$

- **a)** The linac for MAX IV is 300 m long. What is the length of the linac seen from an electron running along the linac with energy 3 GeV?
- **b**) The storage ring is circular with a radius of 84 m. What will the ring look like from a system moving with the same velocity as the 3 GeV electron

$\mathbf{2.3}$

The picture shows a quadrupole magnet consisting of four magnetic cores with coils of wires. In the light grey area between the cores there is pure quadrupole magnetic field.



a) The direction of the current in the upper coil is indicated in the figure. Indicate the direction of the currents in the three other coils.

b) Sketch the magnetic field lines in the light grey area. Mark the direction of the field with arrows.

- c) A beam of positive charges are traveling along the positive z-axis (towards you) through the magnetic field. The beam has a circular cross section when it enters the magnetic field. Which of the cross sections a), b), c) och d) is the most likely when the beam has passed through the magnetic field? Give a motivation. We may assume that the quadrupole is quite short.
- d) Give the equation on the form y = f(x) for the upper and lower curves 1 and 3, and the the equation x = g(y) for the curves 2 and 4.
- e) Determine an analytic expression for the magnetic flow density B(x, y) in the light grey area between the cores if $|B(a, 0)| = B_0$.

<u>Hint</u>: The width, in the z-direction, of the cores is large enough for the field to be considered as two-dimensional. The general solution to Laplace equation $\nabla^2 \Phi(r_c, \phi) = 0$, that is bounded at the origin, is in planar coordinates r_c, ϕ given by

$$\Phi(r_c,\phi) = A_0 + \sum_{n=1}^{\infty} A_n r_c^n \cos(n\phi + \alpha_n)$$

where A_n and α_n are constants. The magnetic flow density is related to the scalar magnetic potential Φ by

$$\boldsymbol{B}(r_c,\phi) = -\mu_0 \nabla \Phi(r_c,\phi)$$

In planar coordinates $\nabla \Phi(r_c, \phi) = \hat{r}_c \frac{\partial \Phi(r_c, \phi)}{\partial r_c} + \hat{\phi} \frac{1}{r_c} \frac{\partial \Phi(r_c, \phi)}{\partial \phi}$ and in cartesian coordinates $\nabla \Phi(x, y) = \hat{x} \frac{\partial \Phi(x, y)}{\partial x} + \hat{y} \frac{\partial \Phi(x, y)}{\partial y}$. Also: $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.

$\mathbf{2.4}$

Assume two beams of electrons running with constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{z}}$. The distance between the beams is d. The beams have circular cross sections and their charge per unit length is ρ_{ℓ} seen from a system at rest.

Determine the force per unit length between the beams as a function of $\beta = v/c$. What happens with the force when $\beta \to 1$?

Basic physics: answers and solutions

S2.1

Electron $\beta = 0.999999985$. Proton ESS $\beta = 0.962$. Proton LHC $\beta = 0.9999999895$.

S2.2

- **a)** 5.2 cm
- **b**) An ellipse with major half axis 84 m and minor half axis 1.45 cm. The minor half axis is parallell and the major axis perpendicular to the direction of motion.

S2.3



- a) See figure.
- **b**) See figure.
- c) Figure in b)
- d) The quadrupole field is given by the n = 2-term in the expansion. This gives $\Phi(r_c, \phi) = A_2 r_c^2 \cos(2\phi + \alpha_2)$. If we let $\alpha_2 = 0$ then

$$\Phi(r_c,\phi) = A_2 r_c^2 \cos(2\phi) = A_2 r_c^2 (\cos^2 \phi - \sin^2 \phi) = A_2 (x^2 - y^2)$$

At the surface of each core $\Phi = \text{constant}$. Thus the four lines are

 $y = \sqrt{x^2 + a^2}$, upper core $y = -\sqrt{x^2 + a^2}$, lower core $x = \sqrt{y^2 + a^2}$, right core $x = -\sqrt{y^2 + a^2}$, left core

where a is the radial distance from the z-axis to the tip of the core.

e)
$$B(x,y) = -\mu_0 \nabla \Phi(x,y) = -\mu_0 A_2(2x,-2y)$$
. In the point $(x,y) = (a,0) B = B_0 \hat{x}$
and then $-\mu_0 A_2 2a = B_0$. This gives $B(x,y) = \frac{B_0}{a}(x,-y)$.

S2.4

For each beam we use a cylindrical coordinate system (r_c, ϕ, z) with z-axis along the axis of the beam. We first determine the electric and magnetic fields from one of the beams (beam A) and then determine the force on the other (beam B).

The electric field from beam A is

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{\boldsymbol{r}}_c \frac{\rho_\ell}{2\pi\varepsilon_0 r_c}$$

where \boldsymbol{r} is outside beam A. Here r_c is the distance to the axis of beam A and $\hat{\boldsymbol{r}}_c$ is the radial unit vector.

Each beam moves with speed v and corresponds to a current $I = \rho_{\ell} v$. According to Ampéres law the magnetic flux density from beam A is

$$oldsymbol{B}(oldsymbol{r}) = \hat{oldsymbol{\phi}} rac{\mu_0 I}{2\pi r_c}$$

where $\hat{\phi}$ is the unit vector in the azimuthal direction.

Let the axis of beam B be located along (x, y, z) = (d, 0, z). Thus $\hat{\mathbf{r}}_c = \hat{\mathbf{x}}$ and $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$. The Lorentz' force per unit length is

$$\boldsymbol{F} = \rho_{\ell} \boldsymbol{E}(d, 0, z) + I \hat{\boldsymbol{z}} \times \boldsymbol{B}(d, 0, z) = \hat{\boldsymbol{x}} \left(\frac{\rho_{\ell}^2}{2\pi\varepsilon_0 d} - v^2 \frac{m u_0 \rho_{\ell}^2}{2\pi d} \right)$$

Since $c^2 = (\mu_0 \varepsilon_0)^2$ the total force is

$$oldsymbol{F} = \hat{oldsymbol{x}} rac{
ho_\ell^2}{2\piarepsilon_0 d} \left(1 - eta^2
ight)$$

3 Synchrotron sources

3.1

In 2012 a new electron storage ring, ASTRID2, was inaugurated at the accelerator laboratory ISA at the University of Aarhus. The ring has the same energy as the first storage ring ASTRID1 but is better optimized for synchrotron radiation and has significantly lower emittance than ASTRID1. Injection into ASTRID2 occurs at full energy (580MeV) from ASTRID1. Below are the main parameters of ASTRID2:

Energy	E = 580 MeV (same as ASTRID1)
Circumference	C = 45.713 m
Radius of curvature for dipoles	$\rho = 1815 \text{ mmm}$
Emittance	$\varepsilon_x = 10 \text{ nm}, \varepsilon_y = 0.5 \text{ nm}$
RF frequency	104.95 MHz
Cavity shunt impedance	$R_s = 1.6 \text{ M}\Omega$
Vacuum pressure	10^{-10} mbar
Stored current	I = 100 mA
	$\Delta p/p = 4 \cdot 10^{-4}$

- a) What is the magnitude of the magnetic flux density, B, in the dipole magnets? Use $\rho = \frac{p}{aB}$.
- **b)** What are the approximate horizontal and vertical size of the beam in the middle of one of the dipole magnets?
- c) What is the maximum size of the beam in the ring, and at which positions in the ring is the horizontal length maximal and the vertical size maximal?
- d) The horizontal quadrupoles in the ring have a field gradient of G = 9.77 T/m. They are $\ell = 22$ cm long. What is their focal length f? (Använd thin lense formula $f = \frac{mc\beta\gamma}{aG\ell}$)
- e) What is the harmonic number $h = f_{\rm Rf}/f_{\rm rev}$? (en bunch i ringen ger h=16)
- f) What is the critical energy of synchrotron radiation from the dipole magnets?
- g) How much power is radiated there?
- **h**) What is the minimum RF voltage required in the cavity?
- i) How much power (time average) must the RF generator at least be able to deliver?

- j) What types of vacuum pumps are suitable? Justify your choice.
- k) In the transfer of the beam line from ASTRID1 to ASTRID2 the current being sent into ASTRID2 is measured. The measurement must be non-destructive. In the case of a pulse with the amplitude of about 10mA and a pulse length of about 100ns. Which type of diagnosis would you recommend for this use? Justify your choice.
- I) In ASTRID1 the electrons are injected with an energy of 100MeV. The beam pipe is made of stainless steel having a thickness of 3 mm. If an electron hits the wall, it can do it with angles of up to 14 degrees with respect to the wall. The density of stainless steel is about 7.8 g/cm³. Can a 100 MeV electron penetrate the wall under these conditions?
- **j**) One of the undulators in ASTRID2 has 30 periods, each with a length of 55 mm. The magnetic flux density at the magnet's surface is 1.1T. With a gap of 23 mm, what is the wavelength in the forward direction ($\Theta_w = 0$) of the first harmonic of the emitted synchrotron radiation?

$\mathbf{3.2}$

At CERN they produce, inter alia, antiprotons (mass as the proton $(M_pc^2 = 938 \text{ MeV})$, but with opposite charge). These antiprotons are collected in an ion trap at low kinetic energy. The antiprotons are produced by letting a proton beam from the PS accelerator hit a metal target. Antiprotons decelerated AD later in the machine (Antiproton Decelerator), and then in the planned ring ELENA. In this exercise, we examine some aspects of ELENA. A sketch of ELENA is seen below. Injection takes place at the top, and there are two locations for extraction. An electron cooler (bottom of the figure) will lower the emittance of the beam.



The Elena lattice functions are shown in the figure below. The solid curve is β_x , the dotted curve is β_y , and the light gray is the horizontal dispersion. Note that β is on the left y-axis and the dispersion D on the right. s = 0 on the x-axis corresponds to the center of the injection line (at the top in the drawing of the ring). The ring circumference is 30.4 m. Here are seven questions concerning ELENA:



a) The AD machine decelerates antiprotons from a momentum of 3.57 GeV/c to a momentum of 100 MeV/c, after which they are injected to ELENA for further

deceleration. What is the kinetic energy of the antiprotons with a momentum of 100 MeV/c?

- b) The particles must decelerate to a kinetic energy of 100 keV in ELENA. The six dipole magnets in ELENA deflects each beam 60° with a radius of 0.927m. What is the strength of the magnetic flux density *B* at extraction and injection, respectively?
- c) In ELENA there are approximately $3 \cdot 10^7$ antiprotons injected and at extraction there is approximately $1.8 \cdot 10^7$ antiprotons. What is the beam current at injection and extraction, respectively?
- d) The electron cooler should lower the beam emittance by letting antiprotons interact with 'cold' electrons running with the same speed as the antiprotons. What is the electron kinetic energy at extraction?
- e) At extracting the aim is that the beam has a horizontal size (s) of 1 mm. $\Delta P/P$ is about 10^{-4} for the cooled beam. Extraction takes place at the positions s = 9 m and s = 24 m, which has the same value of the lattice functions. The values are not so easy to read in the figure, but do your best. What is the horizontal emittance at extraction?
- f) The RF-system in ELENA is to be run at harmonic number $h = f_{\rm Rf}/f_{\rm rev} = 4$. What is the RF-frequency at injection and extraction, respectively?
- g) The bunch length is ~ 1.3 m at the energy at extraction. The extraction is carried out by rapid electrostatic kickers. How quickly must they go from zero to full voltage to extract the beam without loss?

3.3

These tasks all about the accelerator facility at CERN, especially LHC. The LHC is designed to act as a collider, partly with protons partly with heavy ions. We will in this task only look at the injection of protons. The protons have to go through a chain of accelerators, as outlined in the figure to the right.

- a) Duoplasmatron ion source, output energy 100 keV.
- b) RFQ (Radio Frequency Quadrupole) to 750 keV
- c) Linac to 50MeV
- d) PSB (Proton Synchrotron Booster) to 1.4 GeV
- e) PS (Proton Synchrotron) to 25 GeV

- f) SPS (Super Proton Synchrotron) to 450 GeV
- g) LHC (Large Hadron Collider) to 6.5 TeV

The first 4 tasks it is about the PSB and LHC. The most important parameters of such equipment are listed below. The energies in the table are kinetic energies.

PSB:	LHC:
Circumference=157m	Circumference=26658 m
Injection energy: 50 MeV	Injection energy: 450 GeV
Extraction energy: 1.4 GeV	Final energy: 6.5 TeV
Beam: $4 \cdot 10^{12}$ protons	RF frequency: 400 MHz
	Impulse spread $1 \cdot 10^{-4}$
	Normalised emittance $(\epsilon\beta\gamma)$: 4 μ m = 4 mm mrad

Other constants: Protons: $m_p c^2 = 938$ MeV. Electrons: $m_e c^2 = 0.511$ MeV. $c = 299783458 \ {\rm m/s}$



P1: Calculate the beam current in the PBS

- a) at the injection energy
- b) at the extraction energy

P2: This task is about beam structure in the LHC:

- a) How many bunches does the LHC beam has if all RF buckets are filled?
- **b)** What is the bunch spacing, i.e. temporal distance between two consecutive bunches?

c) To allow time for data collection, runtime in sensors etc. one like to have a bunch spacing of at least 25ns. This is achieved by not filling all RF buckets. How many bunches should there be if you want to have at least 25ns spacing?



P3: We now take a look at the beam in the LHC. The ring Twiss parameters (β_x, β_y) and the horizontal dispersion, D_x , in one section is given in the figure above. It is not as easy to read data from, but do your best.

The dispersion curve is the one in the bottom and is only different from zero in a part of the section. Notice that the dispersion is multiplied by 10. For the other two curves, the solid line is for the horizontal beta function β_x , and the dotted for the vertical beta function β_y .

- a) You want a beam pipe whose radius is at least 10 times the maximum beam width (σ_x, σ_y) both at injection and at full energi. What is the smallest diameter the pipe can have?
- **b**) what is the horizontal diameter of the beam at at the maximum energy, where, as you can see, also the dispersion is maximum?

P4: The dipole magnets in the LHC have a magnetic field of 8.3T at 7 TeV.

- a) What is radius of the beam in the dipoles?
- b) How much power is transmitted by synchrotron radiation from a 7 TeV proton beam with a beam current of 530 mA?
- **P5:** Finally, let's look at the LHC's predecessor, LEP. It accelerated electrons and positrons to the energy 105 GeV. The radius of its dipoles was 3096 m.
 - a) What is the critical energy of the emitted synchrotron radiation?
 - b) What effect is the dominant in the attenuation in the lead (Z = 82) for a photon with the critical energy?

c) What is the attenuation coefficient μ_0 in NaI ($\rho = 3.67$ g / cm3) for such a photon?

Synchrotron sources: answers and solutions

S3.1

- a) Use $B\rho = p/q$. If $E \gg m_0 c^2$ then from $E^2 = (m_0 c^2)^2 + p^2 c^2$ we have the general expression, and the relativistic approximation p = E/c, or in more convenient units p = eE[eV]/c, where e is the elementary charge. This gives us the expression $B\rho = E[eV]/(cq[e])$, where q is in units of the elementary charge. Inserting the value for $c (3 \cdot 10^8 \text{ m/s})$, we have the general relativistic approximation $B\rho = 3.33E[\text{GeV}]/q[e]$. This is only valid for $E \gg m_0 c^2$! In this case, with E = 0.58 GeV and q=1, the approximation gives us $B\rho = 1.93$ Tm, and with $\rho = 1.815$ m we have B = 1.06 T.
- **b)** The beam size (one std. dev) due to beam emittance is given by $\sigma_{\varepsilon}(s) = \sqrt{\varepsilon\beta(s)}$, and due to dispersion $\sigma_{\rm D}(s) = D(s)\frac{\Delta p}{p}$. The total size is $\sigma(s) = \sqrt{\sigma_{\varepsilon}(s)^2 + \sigma_{\rm D}(s)^2}$, so using the numbers from ASTRID2 we find $\sigma(s) = 90 \,\mu{\rm m}$ horizontally, with $\beta = 0.5$ m and D(s) = 0.05 m read from the plots of beta and dispersion functions. In the vertical plane the dispersion is zero, so here we find $\sigma(s) = 76 \,\mu{\rm m}$, with $\beta = 11.5$ m
- c) Horizontal: The maximum value of the beta function is 4.7 m. Here, the dispersion is zero, so the beam size is $\sigma_{\varepsilon}(s) = \sqrt{\varepsilon\beta(s)} = 220 \,\mu\text{m}$. The dispersion function is at a maximum between the dipoles, with a value of 0.55m. At this point, beta is 3m which gives a total beam size of 280 nm, so the horizontal beam size is at a maximum between the dipoles, where it is 280 nm. The vertical beam size is clearly at a maximum at the centre of the dipoles, where we found the size in question b to be 76 μ m.
- d) Using g = 9.9 T/m and $B\rho = 1.93$ Tm we find k = 5.06 m⁻². Using the thin lens formula: f = 1/(ks) we find $f = 1/(5.06m^{-2} \cdot 0.22m) = 0.9$ m.
- e) $f_{\rm RF} = 104.95$ MHz and $f_{\rm rev} = c/C = 3 \cdot 10^8 8/45.713 \, {\rm s}^{-1} = 6.563$ MHz, so h=16.

f)
$$\varepsilon_{\rm c} = 7.21 \cdot E [{\rm GeV}]^3 / \rho[{\rm m}] = 237 \, {\rm eV}$$

- g) $\Delta E/\text{turn} = 88.5 \cdot E[\text{GeV}]^4/\rho[\text{m}] = 5.5 \text{ keV}$, so $P[\text{kW}] = I[\text{A}] \cdot \Delta E = 550 \text{ W}$.
- h) $\Delta E = 5.5 \text{ keV/turn}$, so we need at least that accelerating voltage in the cavity. In reality, about 5 - 10 times this number is used, say 50 kV.
- i) With $R_{\rm s} = 1.6 \,\mathrm{M}\Omega$ and $V = 10 \,\mathrm{kV}$, we get $P = V^2/(2R_{\rm s}) = 780 \,\mathrm{W}$ loss in the cavity walls. In addition, the beam needs 550 W (at 100 mA, from question g), so the amplifier must be able to deliver at least a total of ~1350 W. (In reality, you would often design the amplifier for more than this, for example 5 kW.)

- **j**) In a ring producing synchrotron radiation, we want an 100% oil-free UHV vacuum, and since we will need to be able to bake the system to achieve the low pressures needed ($< 10^{-9}$ mb with beam in the machine), all vacuum components (valves, gaskets etc.) must be all-metal, giving low leakage and very low outgassing. Therefore, a good choice will be ion pumps supplemented by Ti sub-limation pumps, eventually also NEG coated tubes where the physical aperture is small, causing poor conductance for pumping by discrete pumps.
- k) For a non-destructive measurement, beam current transformers are ideal for beam currents in the mA range. They are easily made fast enough to show 100ns pulses. A good ferrite toroid with ~ 25 windings will do, and can be home made very cheaply, requiring no amplifiers etc. The above toroid will give you 2 mV/mA in a 50 Ω oscilloscope input.
- 1) For a 100 MeV electron in steel, range/density, R/ρ , is about 20 g/cm². With $\rho \sim 7.8$ g/cm³, R is 26mm. The layer to traverse is 3 mm/sin(14°) = 12.5 mm, so a lot of the electrons will get through the wall.
- **m**) The peak field on axis is $B = B_0/\cosh(\pi g/\lambda_u)$, where g is the gap and λ_u the period length. This gives $B_0 = 0.55$ T. Since K = 2.83, we have for the wavelength $\lambda_{w,n} = \left(\frac{\lambda_u}{2n\gamma^2}\right) \left(1 + \frac{K^2}{2} + \gamma^2 \Theta_0^2\right)$, where n = 1 and $\Theta_0 = 0$, so $\lambda = 107$ nm.

S3.2

- a) From $E = \sqrt{p^2 c^2 + m_0^2 c^4}$ and $E = E_k + m_0 c^2$ we have $E_k = \sqrt{p^2 c^2 + m_0^2 c^4} m_0 c^2$, which, with p = 100 MeV/c and $m_0 c^2 = 938 \text{ MeV}$, gives a kinetic energy $E_k = 5.32 \text{ MeV}$. (Alternative: Use classical expression $E_k = p^2/(2m) = 10000/c^2/(2m) \text{ MeV} = 10000/mc^2 \text{ MeV} = 5.33 \text{ MeV}$)
- **b)** From the expressions in problem 2.1 a), we have the general expression $B\rho = 3.33\sqrt{E_k^2 + 2E_km_0c^2}$, where E_k and m_0c^2 are in GeV, B in T, ρ in m and q in units of the elementary charge. This gives B = 0.36 T at 5.32 MeV and B = 0.05 T at 100 keV.
- c) With antiprotons at 5.32 and 0.1MeV, we can use classical expressions. Since $E_{\rm k} = \frac{1}{2}mv^2$, we have $v = c\sqrt{\frac{2E_{\rm k}}{m_0c^2}}$, which gives $v_{\rm inj} = 3.2 \cdot 10^7 {\rm m/s}$ and $v_{\rm ext} = 4.38 \cdot 10^6 {\rm m/s}$. Using the ring circumference, the revolution frequencies may easily be calculated, and using the number of particles and the elementary charge, we get for the current at injection $I_{\rm inj} = 5.0 \,\mu{\rm A}$, and at extraction $I_{\rm ext} = 0.42 \,\mu{\rm A}$.

- d) Since we can use the classical expression for the kinetic energy, with the same velocity, the kinetic energy is proportional to the particle mass. Therefore, the electron energy at extraction is $10^5 \cdot 0.511/938 = 54$ eV.
- e) For the beam size s we have the expression $\sigma = \sqrt{\varepsilon_x \beta_x + \left(\frac{\Delta p}{p}D\right)^2}$ where β_x and D are functions of position in the ring. By reading β and D from the graph with Twiss functions, we find $\beta_x = 4.5$ m and D = 0.9 m, both at positions 9 and 24 m in the ring. From the above equation, we can then find $\varepsilon_x = 0.22 \,\mu$ m.
- **f)** From question c) we have the revolution frequencies, so to find the RF frequencies we just have to multiply by 4, giving 4.2 MHz and 0.58 MHz at injection and extraction respectively.
- g) To achieve a lossless extraction, the kicker rise time must be shorter than the time from the end of one bunch to the start of the next. Since the ring circumference is 30.4 m, the bunch length is 1.3 m, and there are 4 bunches, we find the distance between bunches to be 6.3 m. With a velocity of $4.38 \cdot 10^6$ m/s at extraction, the time interval is $1.44 \,\mu$ s, which is the maximum acceptable rise time of the kicker.

$\mathbf{S4}$

P1:

- a) First, find the revolution frequency. With 50MeV (non-relativistic), we have $v = c \sqrt{\frac{2E}{m_0 c^2}} = 9.8 \cdot 10^7 \text{ m/s}$. This gives $f_{\text{rev}} = 6.24 \cdot 10^5 \text{Hz}$ and thus $I = Nef_{\text{rev}} = 0.4 \text{ A}$, with N being the number of particles.
- **b)** With 1.4 GeV we must use the general expression $v = c\sqrt{1 \gamma^{-2}}$, which, with $\gamma = (E_{\rm kin} + m_0 c^2)/m_0 c^2$ gives us $v = 2.75 \cdot 10^8$ m/s, or $f_{\rm rev} = 1.75$ MHz. As above, the current is then I = 1.1 A.

P2:

- a) In the LHC, $v \sim c$, both at injection and full energy, and thus $f_{\text{rev}} = O/c$, where O is the circumference of the ring. The number of bunches is then $h = f_{\text{RF}}/f_{\text{rev}} = 35557$.
- b) The bunch spacing is simply the RF wavelength, $1/f_{\rm RF} = 2.5$ ns.
- c) For 25 ns spacing, only every tenth bunch can be filled, since the spacing is 2.5 ns between buckets. This means a total of max 3557 bunches.

- a) First, find the emittance at injection and full energy: The normalized emittance is $e\beta\gamma$, and with $\gamma = 481$ at injection and $\varepsilon = 7643$ at full energy, this means that the emittance is 8.3 nm at 450 GeV and 0.54 nm at 7 TeV. The maximum β (horizontally) is ~380 m, giving beam sizes σ_{β} of 1.8 mm and 0.45 mm resp. The largest value of the dispersion is ~2 m, meaning that σD is $10^{-4} \cdot 0.4\text{m} = 40 \, u\text{m}$, but at that point the β function is only ~180 m, and σ_{β} therefore 1.2 mm, so the beam will be largest at injection energy and at the point where the β function is at a maximum. This size is ~1.8mm, meaning a minimum beam pipe diameter of 36 mm.
- **b)** $D_{\text{max}} \sim 2 \text{ m}$, and with β at that point 180 m, we get $\sigma = \sqrt{\sigma_{\beta}^2 + \sigma_{\text{D}}^2} \sim 0.32 \text{ mm}$ at full energy.

P4:

- a) $B\rho = 3.33 E[\text{GeV}]/q[\text{e}]$ (if $E \gg m_0 c^2$, as is the case for a proton at 7 TeV), so $B\rho = 23310$ Tm, and therefore $\rho = 2808$ m.
- **b)** The power radiated during transverse acceleration is $P = \frac{e^2 c}{6\pi\varepsilon_0} \frac{1}{(mc^2)^2} \frac{E^4}{\rho^2}$, or, in practical units with our usual expression (for electrons): $P_{\rm e}[\rm kW] = I[\rm mA] \cdot 88.5 \cdot E^4/\rho[\rm m] = 4 \cdot 10^{13}$ kW. However, the particle mass enters in fourth power in the denominator of the expression, so a proton will emit a factor of $(m_{\rm e}/m_{\rm p})^4 = 8.85 \cdot 10^{-14}$ of the power emitted by an electron, so $P\rho = 4 \cdot 10^{13} \cdot 8.85 \cdot 10^{-14} = 3.5$ kW.

P5:

- a) For electrons, the critical energy is $2.21E^3/\rho = 826$ keV
- **b**) Compton scattering
- c) $\mu_0/\rho = 0.065 \text{ cm}^2/\text{g}$, $\rho = 3.67 \text{ g/cm}^3$, so $\mu_0 = 0.24 \text{cm}^{-1}$ ($\rho = \text{density}$)