Solutions to Problem Set 1 – Introduction to Accelerator Physics

Problem 1

a) We have

$$E = E_0 + E_k = m_0 \gamma c^2$$

$$E_k = E - E_0 = m_0 \gamma c^2 - m_0 c^2 = m_0 c^2 (\gamma - 1)$$

$$\gamma = \frac{E_k}{m_0 c^2} + 1$$

The rest energy of an electron is $m_e c^2 \approx 0.511$ MeV, which for $E_k = 5$ MeV gives $\gamma = \frac{5}{0.511} + 1 \approx$ 10.785 and for $E_k = 2$ MeV $\gamma = 4.914$.

Use the expression for the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\frac{v_2}{v_1} = \sqrt{\frac{1 - \frac{1}{10.785^2}}{1 - \frac{1}{4.914^2}}} = 1.017 = 1.7\%$$

An electron with $E_k = 5$ MeV is 1.7% faster than an electron with energy $E_k = 2$ MeV.

For protons the rest energy is $m_p c^2 \approx 938$ MeV which for $E_k = 5$ MeV gives $\gamma = \frac{5}{938} + 1 \approx 1.005$ and for $E_k = 2$ MeV $\gamma = 1.002$. This means a proton with $E_k = 5$ MeV is 57.8% faster than a proton with $E_k = 2$ MeV.

b) The velocity is determined by the Lorenz factor. The Lorentz factor is determined by the total energy. Since the rest energy of a proton is larger than the rest mass of an electron a larger kinetic energy is required to reach the same velocity.

Problem 2

We have the Lorentz force $F = q(E + v \times B)$. The energy gain is given by the work done on the particle $W = \int_{r_1}^{r_2} F \cdot dr$. Since the path element dr is always parallel to the velocity vector it is perpendicular to $v \times B$ and thus the magnetic field cannot change the energy of the particle. To

increase the energy of the particle only electric fields can be used, whereas both electric and magnetic fields can be used to change the direction.

Problem 3

a) We have the Lorentz force $F = q(E + v \times B)$ and the expression for circular motion $F = \frac{mv^2}{r}$. This gives

$$\frac{mv^2}{r} = qvB$$

and the velocity

$$v = \frac{qBr}{m}$$

The revolution period is

$$T = \frac{C}{v} = \frac{2\pi r}{v} = \frac{2\pi m}{qB}$$

which gives the angular revolution frequency

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{qB}{m}$$

The angular revolution only depends on the charge, the mass and the magnetic field which all are constant at non-relativistic velocities. As the particles become relativistic the mass increases and the revolution frequency is no longer constant. Therefore the cyclotron can only be used for non-relativistic particles, e.g. protons and ions.

b) We want $\Delta T = k \cdot T_{rf}$ where k is an integer.

From

$$\frac{mv^2}{r} = qvB$$

we have

$$r = \frac{mv}{qB}$$

Since $E = mc^2$ this can be written

$$r = \frac{mc^2 v}{qc^2 B} = \frac{v}{qc^2 B}E$$

which gives

$$E = \frac{r}{v}qc^2B$$

As in a) we have

$$T = \frac{2\pi r}{v}$$

which gives

$$E = \frac{T}{2\pi} qc^2 B$$

The energy gain is then

$$\Delta E = E_{i+1} - E_i = \frac{T_{i+1}}{2\pi} qc^2 B - \frac{T_i}{2\pi} qc^2 B = \frac{qc^2 B}{2\pi} \Delta T$$

With the condition $\Delta T = k \cdot T_{rf}$ we get

$$\Delta E = \frac{qc^2 B}{2\pi} \Delta T = \frac{qc^2 B}{2\pi} k \cdot T_{rf} = k \frac{qc^2 B}{2\pi} \frac{1}{f_{rf}} = k \frac{qc^2 B}{\omega_{rf}}$$

The required energy gain increases with k and B, but decreases with ω_{rf}.

c) We have

$$r = \frac{mv}{qB} = \frac{p}{qB}$$

We need the relativistic relation between energy and momentum:

$$p = mv = \gamma m_o v = \gamma m_o \beta c$$

$$pc = \gamma m_o \beta c^2 = \gamma m_o \sqrt{1 - \frac{1}{\gamma^2}} c^2 = \sqrt{\gamma^2 - 1} m_o c^2$$

$$p^2 c^2 = (\gamma^2 - 1) m_o^2 c^4 = \gamma^2 m_o^2 c^4 - m_o^2 c^4 = E^2 - m_o^2 c^4$$

$$E^2 = p^2 c^2 + m_o^2 c^4$$

This gives

$$r = \frac{p}{qB} = \frac{\sqrt{E^2 - m_0^2 c^4}}{qBc} \approx \frac{E}{qBc}$$

since $E^2 \gg m_0^2 c^4$. This gives

$$r = \frac{(25 + 0.511) \cdot 10^6}{1.2 \cdot c} \approx 0.07 \, m$$

Problem 4

We have the length of a drift tube $l_n = v_n \cdot t_n$. The condition for acceleration is $t_n = \frac{T_{RF}}{2} = \frac{1}{2}\frac{1}{f_{RF}}$ since then the electron only sees an accelerating field. This gives $l_n = \frac{v_n}{2f_{RF}}$. We have $v_n = \beta c$ which gives

$$l_n = \frac{v_n}{2f_{RF}} = \frac{\beta c}{2f_{RF}} = \frac{c}{2f_{RF}}\sqrt{1 - \frac{1}{\gamma^2}}$$

With $E = m_0 \gamma c^2$ this gives $\gamma = \frac{E}{m_0 c^2}$ and

$$l_n = \frac{c}{2f_{RF}}\sqrt{1 - \left(\frac{m_0 c^2}{E}\right)^2}$$

where $E = E_0 + \Delta E$.

For protons we have $E_0 = 938$ MeV, $\Delta E_1 = 0.1 + 1 = 1.1$ MeV (since the starting kinetic energy is 100 keV) and $\Delta E_5 = 5.1$ MeV. This gives

$$l_1 = \frac{c}{2 \cdot 7e6}\sqrt{1 - \left(\frac{938e6}{938e6 + 1.1e6}\right)^2} \approx 1.04 \; m$$

and $l_5 \approx 2.24$ m.

For electrons we have $W_0 = 0.511$ MeV, $\Delta W_1 = 1.1$ MeV and $\Delta W_5 = 5.1$ MeV which gives $l_1 \approx 20.3$ m and $l_5 \approx 21.3$ m.

For electrons the drift tubes have to be very long since the electrons become relativistic at low energy. This means the Wideroe accelerator is more useful for protons or low energy electrons.

Problem 5

a) We use $n = 1$, $I = 400$ A, $\mu_0 = 4\pi \cdot 10^{-7}$ N/A^2 and

$$B = \frac{nI\mu_0}{h}.$$

This gives

$$h = \frac{nI\mu_0}{B} = 5.06 \; mm.$$

b) We have $L = 1$ m, $S = 5$ mm^2, $\rho = 1.7 \cdot 10^{-2}$ Ω mm^2/m. We have the power

$$P = UI = RI^2,$$

and the resistance

$$R = \rho\frac{L}{S} = 1.7 \cdot 10^{-2}\frac{1}{5} = 0.0034 \; \Omega.$$

This give the power

$$P = RI^2 = 0.0034 \cdot 400^2 = 544 \, W.$$

For $n = 20$ to get the same field we need

$$I = \frac{Bh}{n\mu_0} = 20 \, A.$$

This gives

$$R = \rho \frac{L}{S} = 1.7 \cdot 10^{-2} \frac{20}{5} = 0.068 \, \Omega,$$

and

$$P = RI^2 = 0.068 \cdot 20^2 = 27.2 \, W.$$

Problem 6

The full turn is 2π, so the bending angle of one dipole is $\theta = \frac{2\pi}{24} = 15°$. Since the length of the magnet is much smaller than the bending radius the small angle approximation is valid giving

$$R = \frac{L}{\theta} = \frac{1 \cdot 24}{2\pi} \approx 3.820 \text{ m.}$$

For a dipole we have

$$\frac{e}{p} B = \frac{1}{R}$$

For $E_k = 1.5$ GeV we have $\gamma \approx 2936$, which gives $\frac{p}{e} = \frac{m_e \gamma c}{e} \approx \frac{8.0185 \cdot 10^{-19}}{1.602 \cdot 10^{-19}} \approx 5$. This results in $B = 1.31$ T.

Problem 7

The number of possible bunches in the ring can be calculated as RF frequency divided by the revolution frequency since after one bunch has passed the cavity, the cavity has more cycles in which it can accelerate other bunches before the same bunch arrives again.

$$h = \frac{f_{rf}}{f_0} = f_{rf} T_0 = f_{rf} \frac{C}{c} = \frac{100 \, MHz}{0.568 \, MHz} = 176.$$

The total charge can be calculated as the current times the revolution time since the current describes the charge passing one point in the machine per second and the revolution time is the time it takes for the whole beam to pass that point.

$$Q_{tot} = IT_0 = 0.5 \cdot \frac{528}{c} = 880 \, nC.$$

The bunch charge is then

$$Q_b = \frac{880}{176} = 5 \ nC.$$

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