

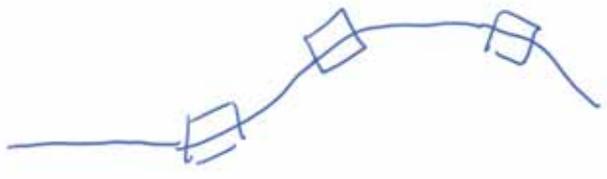
NPAS  
2017  
M

SVERKER  
WERIN

# How does an electron move?

- How does the  $e^-$  travel and where does it end up?
  - What happens over many turns in a ring?
  - How does a complete beam behave?
- 
- A single electron in a magnetic field.
  - Matrices to image the accelerator
  - Phase space to image  $e^-$  motion
  - A beam of particles
  - emittance, phase space,  $\beta$ -function to image a beam of particles
  - Something on the dynamics of the beam of particles

# Introduction



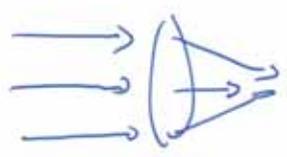
bend



focus

Deviation  $x, x'$

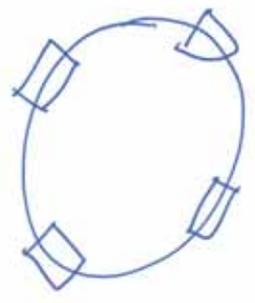
Ideal orbit



Light optics  
 focal length, chromaticity  
 (aberrations)



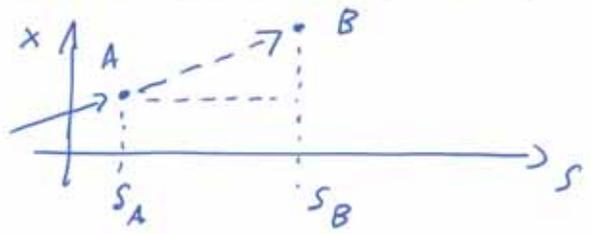
Individual  $\vec{c} \rightarrow$  beam  
 cross section  
 size & divergence  
 phase space



Ring  
 Oscillations

orbit length  
 Errors / imperfections  
 Stability / resonances

# Intro - Matrices

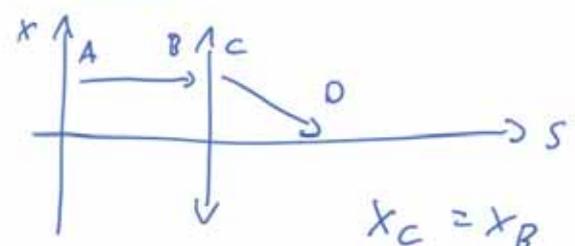


A:  $(x, x')$

B:  $(x_B, x'_B) = (x + x'(s_B - s_A), x')$

OR  $\begin{pmatrix} x_B \\ x'_B \end{pmatrix} = \begin{pmatrix} 1 & s_B - s_A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$

## A Lense



A → B as above  
C → D as above

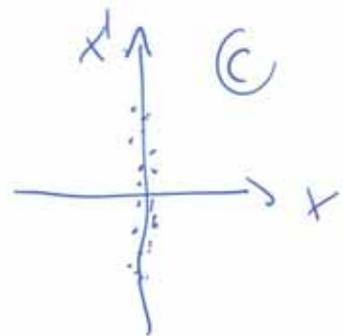
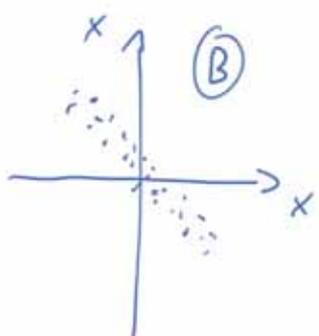
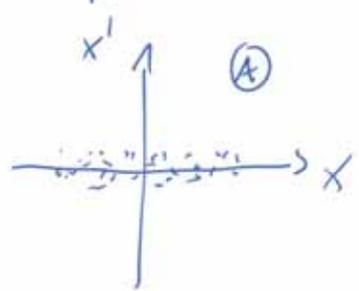
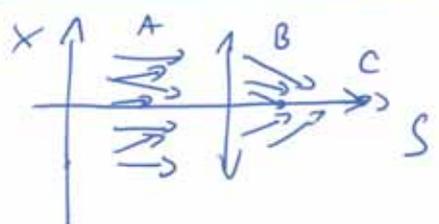
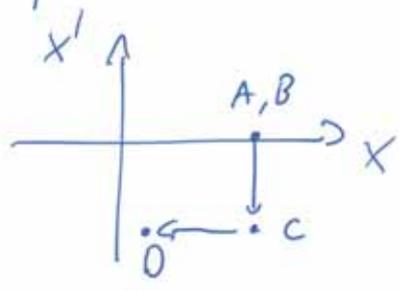
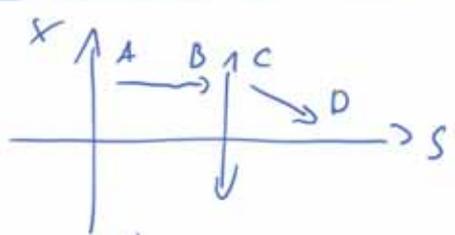
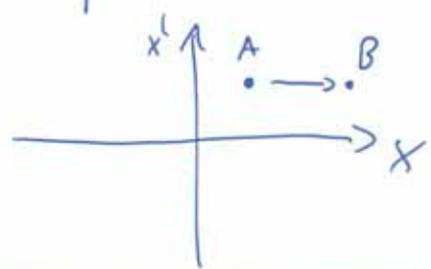
$x_C = x_B$

$x'_C = x'_B + \alpha = x'_B + kx_B$

$\begin{pmatrix} x_C \\ x'_C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} x_B \\ x'_B \end{pmatrix}$

$\begin{pmatrix} x_D \\ x'_D \end{pmatrix} = \begin{pmatrix} 1 & l_{CD} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & l_{AB} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_A \\ x'_A \end{pmatrix}}_{\begin{pmatrix} x_B \\ x'_B \end{pmatrix}} \underbrace{\hspace{10em}}_{\begin{pmatrix} x_C \\ x'_C \end{pmatrix}}$

# Intro - phase space

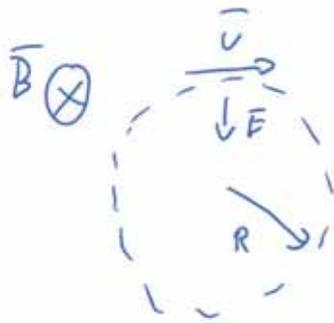


# Equations of motion - Lorentz eq

$$\begin{cases} \frac{d}{dt} (\gamma m_0 c^2) = e \bar{v} \cdot \bar{E} \\ \frac{d}{dt} (\gamma m_0 \bar{v}) = \bar{F} = e (\bar{E} + \bar{v} \times \bar{B}) \end{cases}$$

$$B \approx 1 T$$

$$v \approx c \Rightarrow E \sim 3 \cdot 10^8 \text{ V/m}$$



$$F_{\text{mag}} = e v B$$

$$F_{\text{cp}} = \frac{m v^2}{R}$$

$$\Rightarrow R = \frac{m v}{e B}$$

$$m = \gamma m_0 \quad p = m v \quad v = \beta \cdot c = c \sqrt{1 - \frac{1}{\gamma^2}}$$

## Other B?

$$\bar{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

Treat separately

Assume = 0

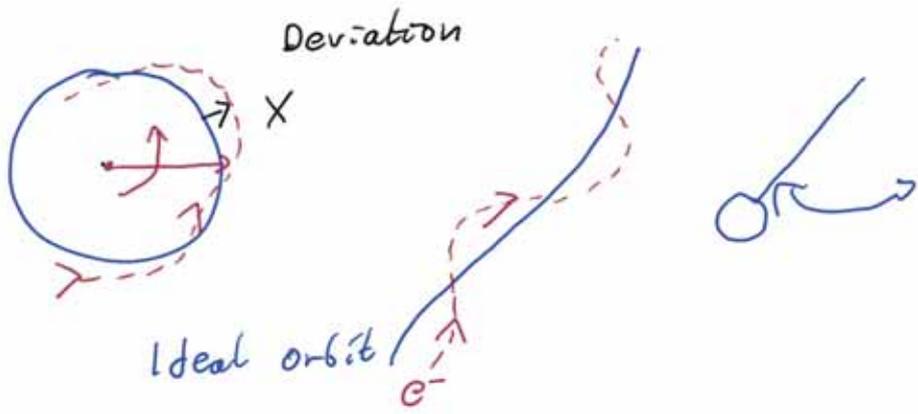
$$B_z(x) = B_{z0} + \frac{dB_z}{dx} x + \frac{1}{2!} \frac{d^2 B_z}{dx^2} x^2 + \frac{1}{3!} \frac{d^3 B_z}{dx^3} x^3 + \dots$$

↑  
Dipole

↑  
Quadrupole

↑  
sextupole

↑  
octupole



$$X'' + k(s)X = 0$$

oscillator

$k(s)$  the magnets along the accelerator

$k = 0$  No magnets

$k = \frac{1}{R^2}$  Dipole / Bending magnet  
 R radius  $B = B_0$

$k = k$  Quadrupole.  $B = gx = \frac{mV}{e} \cdot k$

## Motion in a magnet

$$x''(s) - k(s)x(s) = 0$$

$$\text{Solution: } \begin{cases} x(s) = A \cos \sqrt{k} \cdot s + B \sin \sqrt{k} \cdot s \\ x'(s) = -A\sqrt{k} \sin \sqrt{k} \cdot s + B\sqrt{k} \cos \sqrt{k} \cdot s \end{cases}$$

$$\text{Start in } x(0) = x_0, \quad x'(0) = x'_0 \\ \Rightarrow A \text{ \& } B$$

$$x(s) = x_0 \cos \sqrt{k} s + \frac{x'_0}{\sqrt{k}} \sin \sqrt{k} s$$

$$x'(s) = -\sqrt{k} \sin \sqrt{k} s + \cos \sqrt{k} s$$

$$\text{Re-write: } \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos \sqrt{k} \cdot s & \frac{1}{\sqrt{k}} \sin \sqrt{k} \cdot s \\ -\sqrt{k} \sin \sqrt{k} \cdot s & \cos \sqrt{k} \cdot s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

!  $k < 0 \quad \sqrt{k}$   
 •  $\sin \rightsquigarrow \sinh$

Test  $k = 0$

$$\Rightarrow \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

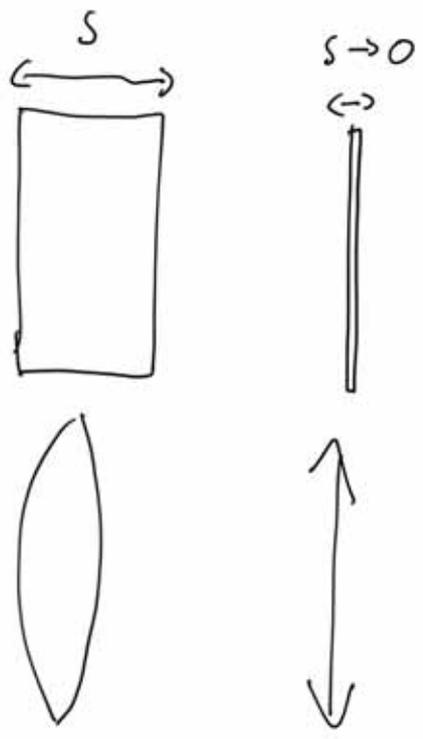
No dipole  
 No quad  $\Rightarrow$  Drift

# Thin lense

$$\underline{M} = \begin{pmatrix} \cos \sqrt{k} s & \frac{1}{\sqrt{k}} \sin \sqrt{k} s \\ -\sqrt{k} \sin \sqrt{k} s & \cos \sqrt{k} s \end{pmatrix} \quad \begin{matrix} s \rightarrow 0 \\ s \cdot k \text{ const.} \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & \frac{1}{\sqrt{k}} \sqrt{k} s \\ -\sqrt{k} \cdot \sqrt{k} s & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -k s & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$



same focal strength/length.

# Dipole

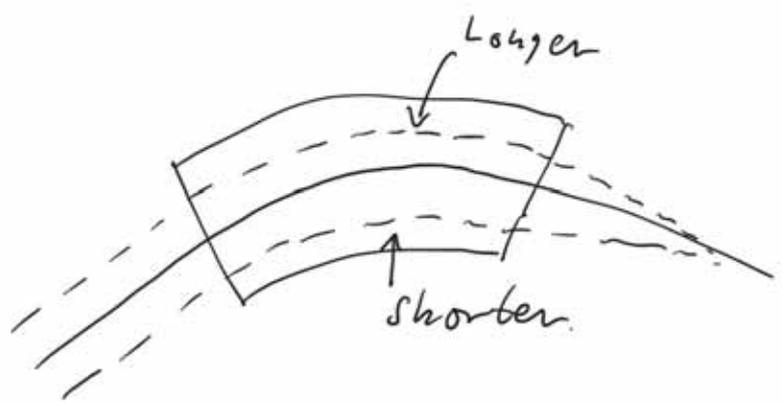
$$k=0 \quad R \neq \infty$$

Replace  $-k \Rightarrow \frac{1}{R^2}$

assume for now  $\Delta p = 0$

$$\Rightarrow \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos \frac{s}{R} & R \sin \frac{s}{R} \\ -\frac{1}{R} \sin \frac{s}{R} & \cos \frac{s}{R} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Looks like a quad  
Focus!



sector mag.

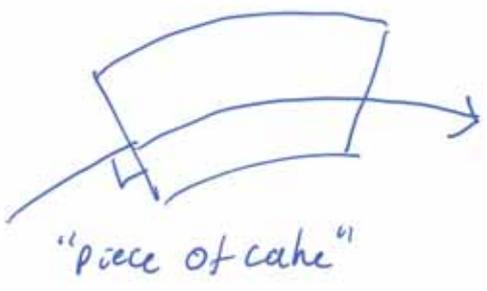
The other plane  $R = \infty$

$$\Rightarrow \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Drift.

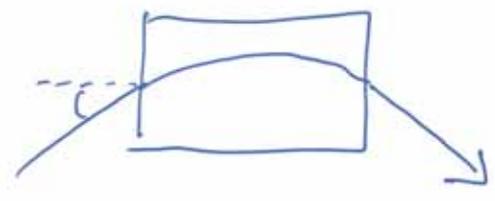
# Rectangular and sector magnet

Sector

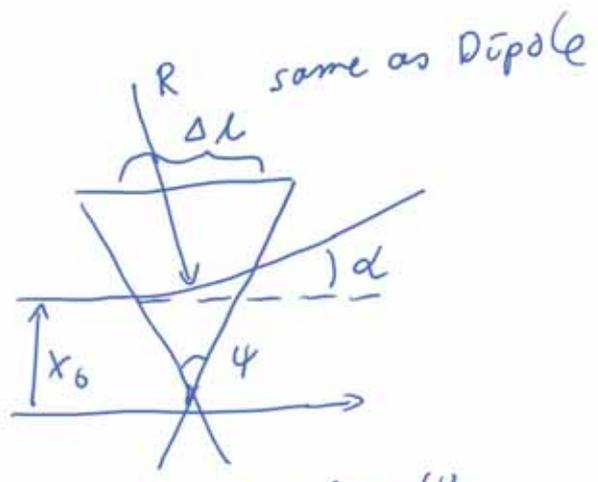
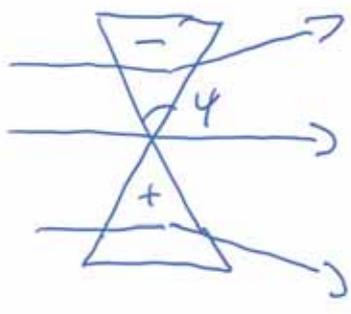
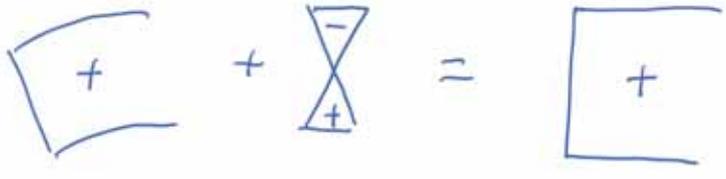


"piece of cake"

Rectangular



JUMP



same as Dipole

$$\Delta L = x_0 \tan \psi$$

$$d = \frac{\Delta L}{R} = x_0 \frac{\tan \psi}{R}$$

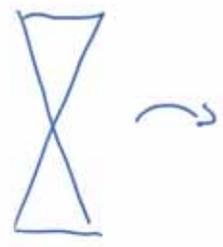
$$\Rightarrow \begin{cases} x_1 = x_0 \\ x'_1 = x'_0 + x_0 \frac{\tan \psi}{R} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\tan \psi}{R} & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Defoc. thin  
lense

comb.

the vertical plane



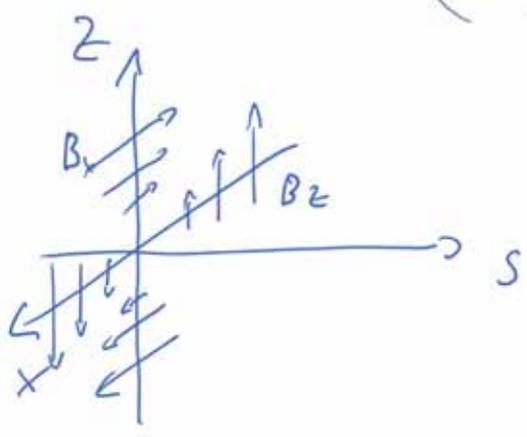
A thin lens where the field increases.

$B_w$  uniform

$$B_z = -x B_w$$

$$\nabla \times B = 0 \Rightarrow \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z} \Rightarrow B_x = -z B_w$$

$$\Rightarrow \begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\tan \psi}{R} & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z'_0 \end{pmatrix}$$



Focusing.

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ \frac{\tan \psi}{R} & 1 \end{pmatrix} \underset{M_{\text{sect}}}{=} \begin{pmatrix} 1 & 0 \\ \frac{\tan \psi}{R} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{\tan \psi}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\tan \psi}{R} & 1 \end{pmatrix}$$

# Dispersion

The path of an  $e^-$  with a different energy.

jump  $\frac{1}{R^2} X = \frac{1}{R} \frac{\Delta P}{P_0}$   $\left( X'' + \frac{1}{R^2} X = 0 \right)$

$$X(s) = X_{HOMO} + X_{PART} = X(s) + X_D(s)$$

$$D(s) \equiv \frac{x_D(s)}{\Delta P / P_0} \Rightarrow D'' + \frac{1}{R^2} D = \frac{1}{R}$$

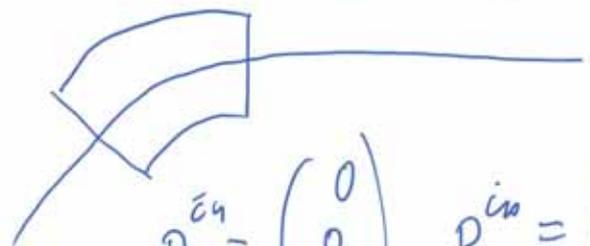
$$\begin{aligned} D(0) &= 0 \\ D'(0) &= 0 \end{aligned}$$

$$\begin{cases} D(s) = R \left( 1 - \cos \frac{s}{R} \right) \\ D'(s) = s \sin \frac{s}{R} \end{cases} \quad \begin{cases} X_D(s) = \frac{\Delta P}{P_0} D(s) \\ X_D'(s) = \frac{\Delta P}{P_0} D'(s) \end{cases}$$

$$\begin{pmatrix} X \\ X' \\ \frac{\Delta P}{P_0} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & D \\ m_{21} & m_{22} & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_0' \\ \frac{\Delta P}{P_0} \end{pmatrix}$$

# Example dispersion

JUMP



$$P_1^{\text{in}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$P_2^{\text{in}} = \begin{pmatrix} 0 \\ 0 \\ \Delta P_2 / P_0 \end{pmatrix}$$

$$P_1^{\text{out}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$P_2^{\text{out}} = \begin{pmatrix} R(1 - \cos \frac{s}{R}) \frac{\Delta P_2}{P_0} \\ S \sin \left( \frac{s}{R} \right) \frac{\Delta P_2}{P_0} \\ \Delta P_2 / P_0 \end{pmatrix}$$



If all  $e^-$  have an energy deviation, this becomes the new orbit.

② can a quadrupole,  $R=0$ , influence the dispersion?

Beta function, TWISS / A solution  
 Look at a whole beam.

$$x''(s) - k(s)x(s) = 0 \quad k(s) = k(s+L)$$

Hill's eq.

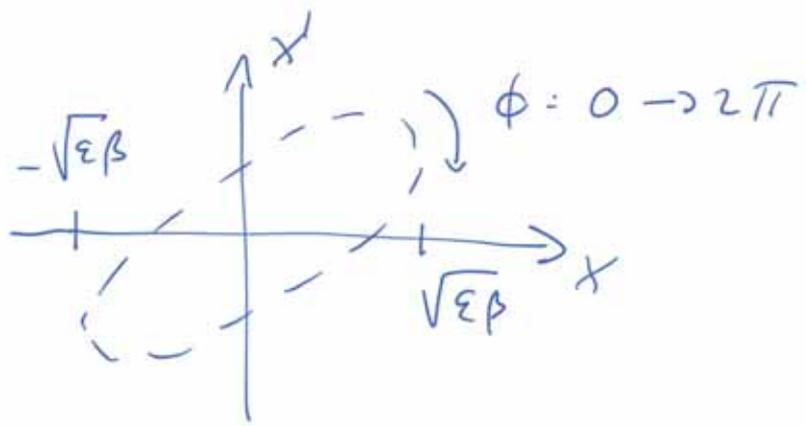
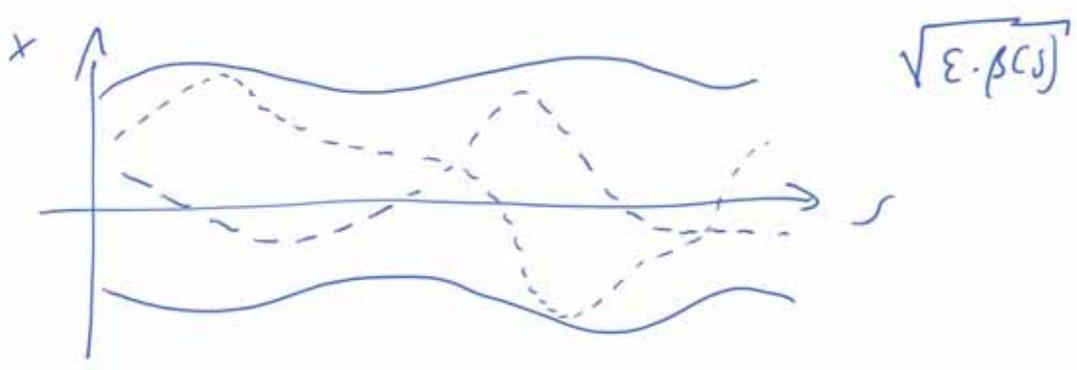
Assume:  $x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos[\psi(s) + \phi]$

$\sqrt{\epsilon}$ : ampl.  
 $\sqrt{\beta(s)}$ : ampl. scaling  
 $\psi(s)$ : phase of osc.  
 $\phi$ : initial phase

$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}} \left[ \alpha(s) \cos[\ ] + s \sin[\ ] \right]$$

$$\begin{cases} \beta(s) \equiv \frac{1}{k(s)} \\ \alpha(s) \equiv -\frac{\beta'}{2} \\ \gamma(s) \equiv \frac{1 + \alpha^2}{\beta} \end{cases}$$

TWISS PARAMETERS



What is  $\bar{\omega} \beta(s)$ ?  
Individual  $e^-$   
 $\leftrightarrow$   
Beam of  $e^-$

# Emittance and Liouville

JOMP

Use (1)  $x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi)$

(2)  $x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}} \left[ \frac{\cos}{2} + \beta \psi' \sin \right]$

Rewrite  $\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\epsilon \beta(s)}}$

Insert in (2)  $\sin(\psi(s) + \phi) =$

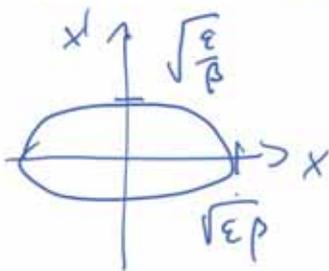
$\Rightarrow \sin^2 + \cos^2 = 1 = \frac{1}{\epsilon} \left( \frac{x^2}{\beta} + \left( \frac{d(s)}{\sqrt{\beta}} x + \sqrt{\beta} x' \right)^2 \right)$

$\Rightarrow \epsilon = \gamma(s) x^2 + 2\alpha(s) x x' + \beta(s) x'^2$

↑  
Ellipse with axes  $x$  &  $x'$   
and area =  $\pi \cdot \epsilon$

↑  
constant!

Simplified  $\alpha = 0$



Area =  $\pi \cdot \text{half axis} \cdot \text{half axis} =$   
 $= \pi \sqrt{\frac{\epsilon}{\beta}} \cdot \sqrt{\epsilon \beta} = \pi \epsilon$

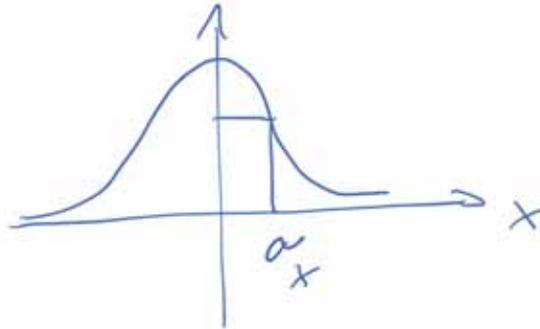
A particle on the ellipse  
will always be on an  
equally big ellipse.

# Beam size

How to define the size?

Gaussian beam

$$\rho(x, z) = \frac{N_0}{2\pi\sigma_x\sigma_z} e^{\left(-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\right)}$$



$$\sigma_x = \sqrt{\epsilon \beta c s}$$

↗ could be defined differently!

② How wide does a vacuum chamber need to be?

# Evolution of the $\beta$ -function

PREDICT  $\beta$   
AT ANOTHER POINT

If the beam is described by  $\epsilon$  &  $\beta$ ,  
How to find  $\beta(s)$ ?

$$\bar{x}_0 = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad \bar{x}_0^T = (x_0 \ x'_0)$$

$$\underline{B}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

write  $\bar{x}_0^T \underline{B}_0^{-1} \bar{x}_0 = (x_0 \ x'_0) \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} =$

$$= \gamma_0 x_0^2 + 2\alpha_0 x_0 x'_0 + \beta_0 x'^2 = \epsilon$$

From before.  
constant everywhere!

jump

AG pos. s

$$\bar{x}_1 = \underline{M} \cdot \bar{x}_0$$

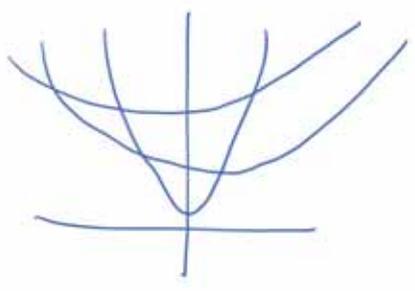
as  $\epsilon = \bar{x}_0^T \underline{B}_0^{-1} \bar{x}_0 = \dots = \bar{x}_1^T (\underline{M} \underline{B}_0 \underline{M}^T)^{-1} \bar{x}_1 =$

$$= \bar{x}_1^T \underline{B}_1 \bar{x}_1$$

$$\Rightarrow \underline{B}_1 = \underline{M} \underline{B}_0 \underline{M}^T$$

Example  $\underline{M}_{drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$  start at  $\beta'_0 = 0 \Rightarrow \alpha_0 = 0$

$$\underline{B}_1 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & s \\ s & 1/\beta_0 \end{pmatrix}$$



$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Predict  $\beta$  and  $\alpha$  at a later point

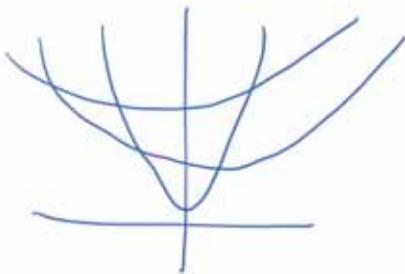
$$\underline{B}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

TRANSFER MATRIX

$$\underline{B}_1 = \underline{M} \underline{B}_0 \underline{M}^T$$

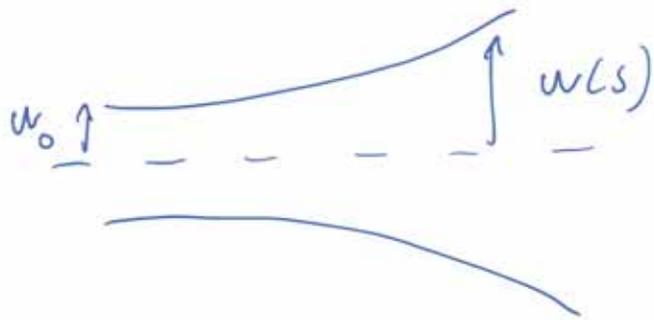
Example  $\underline{M}_{drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$  start at  $\beta' = 0 \Rightarrow \alpha = 0$

$$\underline{B}_1 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & \frac{s}{\beta_0} \\ \frac{s}{\beta_0} & \frac{1}{\beta_0} \end{pmatrix}$$



$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

# Gaussian light beam



$$w(s) = w_0 \sqrt{1 + \left(\frac{s \lambda}{\pi w_0^2}\right)^2}$$

$$w^2 = w_0^2 \left(1 + \left(\frac{s \lambda}{\pi w_0^2}\right)^2\right) = w_0^2 + \left(\frac{s \lambda}{\pi}\right)^2 \frac{1}{w_0^2} =$$

$$= w_0^2 + \left(\frac{\lambda}{\pi}\right)^2 \frac{s^2}{w_0^2}$$

$$\text{size} \sim \sqrt{\beta} \quad \beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$



# Matrix from TWEIN

To learn more, turn the problem around.  
 Not practical, but useful for analysis.

JUMP

$$\begin{cases} x(s) = \sqrt{\varepsilon \beta(s)} \cos(\psi(s) + \phi) = \\ \quad = \sqrt{\varepsilon \beta(s)} [\cos \psi \cos \phi - \sin \psi \sin \phi] \\ x'(s) = \dots \end{cases}$$

Assume initial conditions:

$$\begin{aligned} x(0) &= x_0 & \beta(0) &= \beta_0 & \psi(0) &= 0 \\ x'(0) &= x'_0 & \alpha(0) &= \alpha_0 & & \end{aligned}$$

$$\Rightarrow \begin{cases} x(0) = x_0 = \sqrt{\varepsilon \beta_0} \cos \phi \\ x'(0) = x'_0 = \dots \end{cases}$$

$$\begin{cases} x(s) = \sqrt{\frac{\beta(s)}{\beta_0}} [\cos \psi(s) + \alpha_0 \sin \psi] x_0 + \sqrt{\frac{\beta(s)}{\beta_0}} \sin \psi(s) x'_0 \\ x'(s) = \dots \end{cases}$$

$$\Rightarrow \bar{X} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}} [\cos \psi + \alpha_0 \sin \psi] & \sqrt{\beta \beta_0} \sin \psi \\ \frac{(\alpha_0 - \alpha) \cos \psi - (1 - \alpha \alpha_0) \sin \psi}{\sqrt{\beta \beta_0}} & \sqrt{\frac{\beta}{\beta_0}} (\cos \psi - \alpha_0 \sin \psi) \end{pmatrix} \bar{X}_0$$

↑  
 TRANSFER MATRIX.

cont.

Assume one turn

$$\beta(s=L) = \beta_0$$

Start as a symmetry point  $\Rightarrow d_0 = d = 0$

$$\Rightarrow \underline{M} = \begin{pmatrix} \cos \psi & \beta_0 \sin \psi \\ -\frac{\sin \psi}{\beta_0} & \cos \psi \end{pmatrix}$$

$$\Rightarrow \beta_0 = \sqrt{\frac{-m_{12}}{m_{21}}}$$

Fixes the  $\beta$ -function!

$$\psi(s=L) = \arccos(m_{11})$$

$\uparrow$  But only fraction of the angle

$$\text{Define } \psi(s=L) = 2\pi Q$$

$\uparrow$  Tune  
Number of oscillations over one turn in the machine.

# Repetition

(I)

The model  
("idea")

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} M_{TRX} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Describe particles  
in phase space



The physics  
the magnets  
The system (and small deviations)

Lorentz  $\vec{F} = e(\vec{v} \times \vec{B})$   
 $\vec{B} = \dots$



The physics revisited

$$x'' + k(s)x = 0$$

↑  
Dip.  
Quad.

$\Rightarrow \underline{M}_{Dip} \quad \underline{M}_{Quad}$

Cont

Build the model

when  $\frac{\Delta p}{p} \neq 0$  adjust M

(11)

$\Rightarrow$  dispersion  $D, X_D$

Another description of the particle

TWISS  $\beta, \alpha, \gamma$

$\hookrightarrow$  The way to start  
look at a beam.

Liouville's theorem  
and emittance

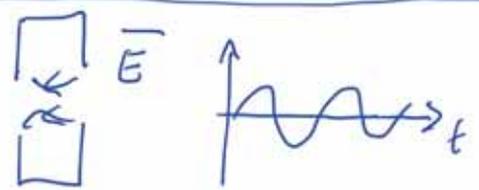
concepts  
important for  
beams!

How to find TWISS around  
the machine.

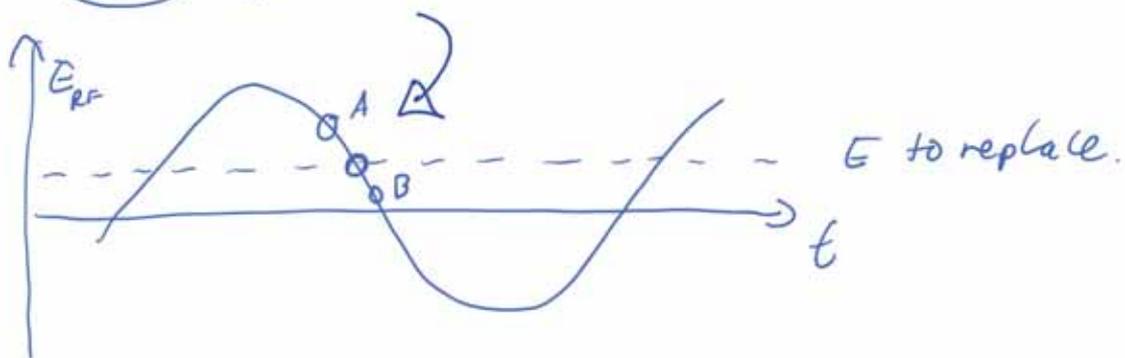
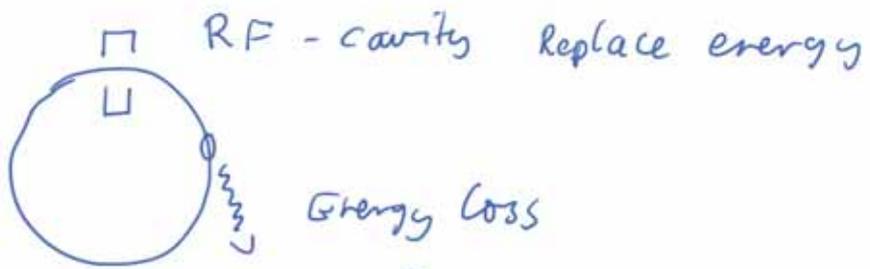
... and write M using TWISS ( $\beta, \alpha, \gamma$ )



# Longitudinal motion



$$Circumf = n \lambda = n \frac{c}{f}$$

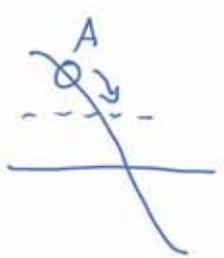


at A gets more energy

⇓  
Larger Longer orbit

⇓  
Arrives later next turn

⇓  
gets less energy

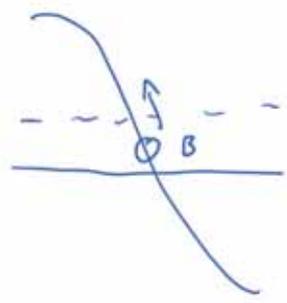


at B gets less energy

⇓  
smaller shorter orbit

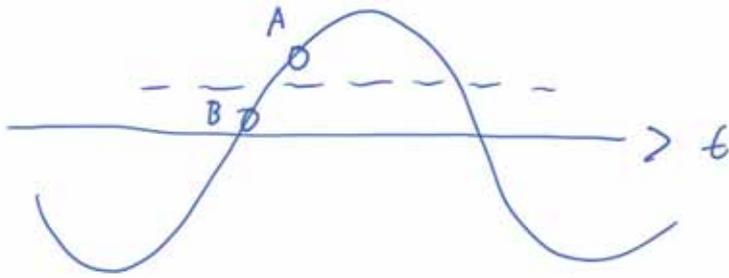
⇓  
Arrives earlier next turn

⇓  
gets more energy



⇒ Synchrotron oscillation

Cont.



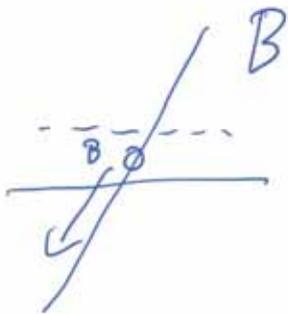
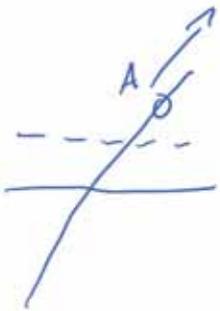
A comes later

gains more energy

comes later

gains more energy

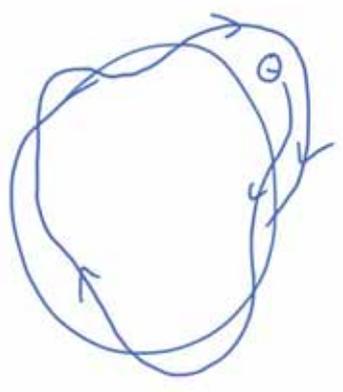
UNSTABLE



v.v

This is not a stable phase!

# Tune & optical resonances

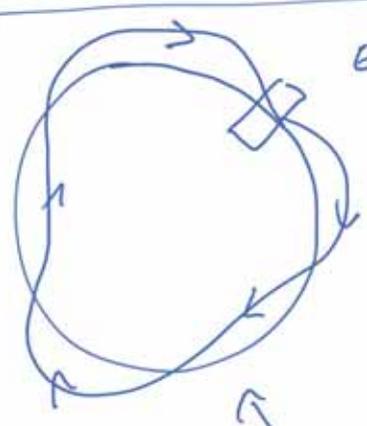


oscillation around C.O.  
 $\beta$ -tron oscillation.

amplitude  $\sqrt{\epsilon p(s)}$

osc. phase  $\psi(s)$

In one turn  $\Delta\psi = 2\pi Q$



Error

- a small magnet

- a small error in an existing magnet  
 (strength, displacement, rotation, extra type)

Error once per turn

New Closed orbit

New tune  $Q$

Resonance.

# Hill's eq with an error

JUMP

prev.  $x'' - k(s)x = 0$

then  $x'' - k(s)x = \frac{1}{R} \frac{\Delta p}{p_0}$

→ gave us dispersed new orbit.

Now  $x'' - k(s)x = F(s)$  ← an error at position  $s$ .

$$\Rightarrow X_{err}(s) = \dots = \frac{\sqrt{\beta(s)}}{2 \sin(\pi Q)} \int_s^{s+L} \sqrt{\beta(t)} F(t) \cos[\psi(t) - \psi(s) - \pi Q] dt$$

$$= \frac{\sqrt{\beta(s)}}{2 \sin(\pi Q)} \sqrt{\beta(s_{kitch})} F(s_{kitch}) \cos[\psi(s_{kitch}) - \psi(s) - \pi Q] L$$

The new orbit will scale as the  $\beta$ -fcn

if  $Q = \text{integ}$   
 $x \rightarrow \infty$

The new orbit ampl. will osc. as the  $e^-$  motion

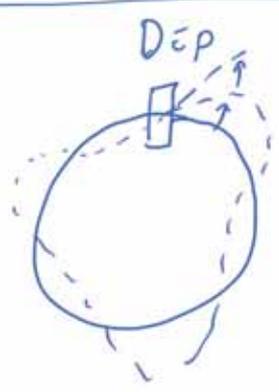


To treat general magnetic errors other methods are necessary



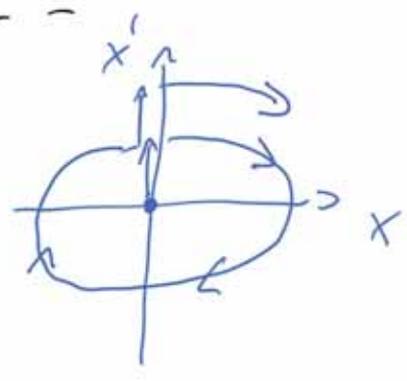
In simulation. take a perfect machine and introduce a small corrector magnet and study the difference.

# Resonances at different $Q$ ?

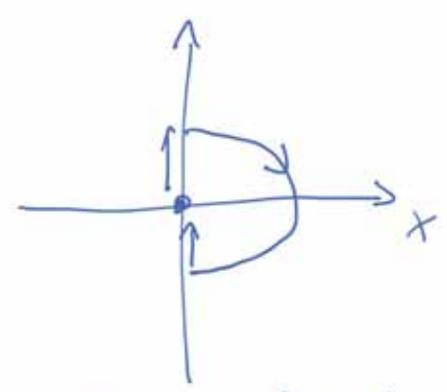


if  $Q$  is an integer

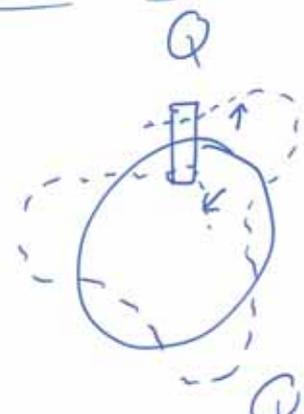
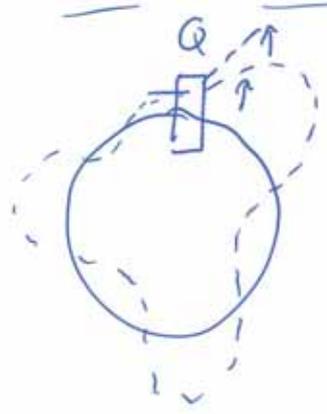
Jump



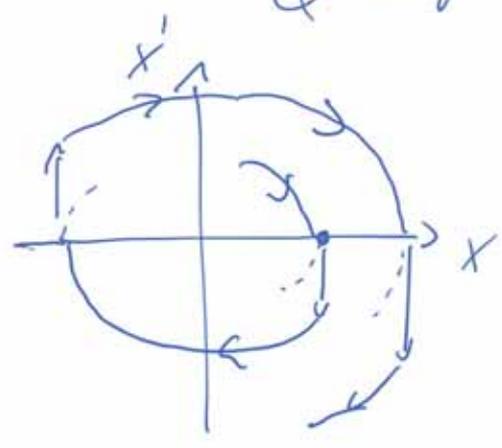
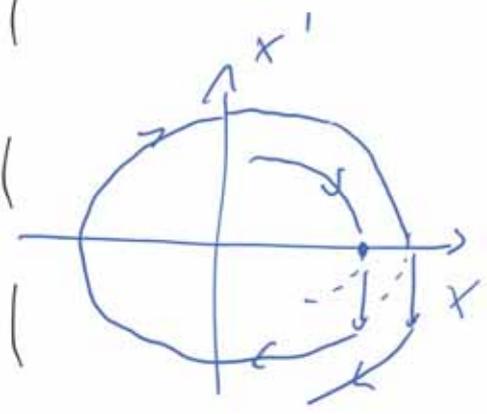
but



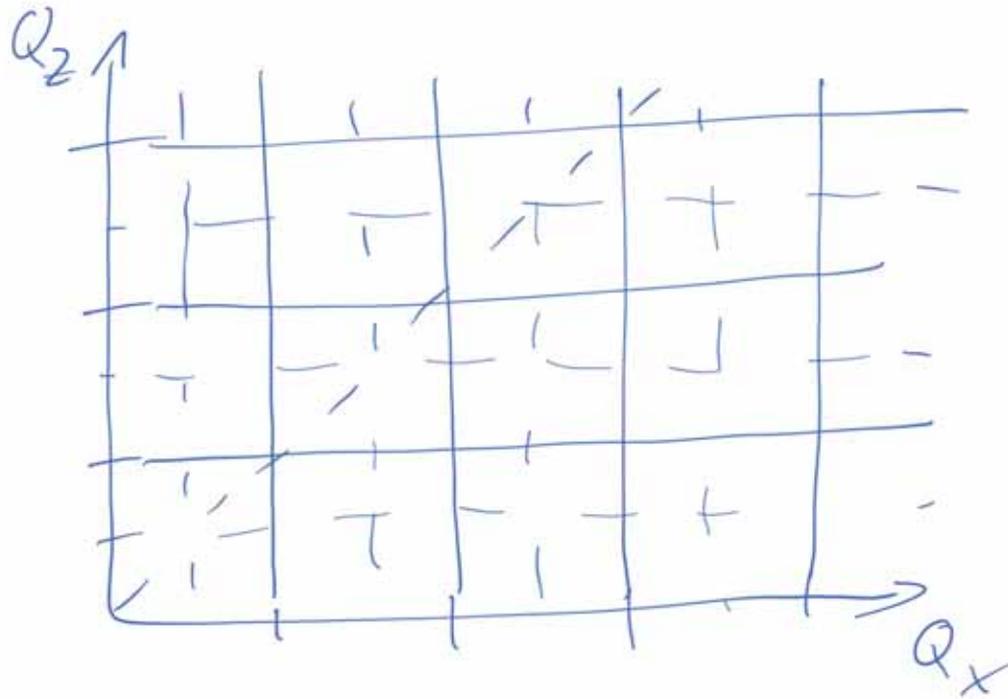
$Q$  half integer.



$Q$  half integer



# Tune diagram



Can get tricky!

# JUMP - - -

## Quadrupole error

$$k = k_0 + \Delta k$$

$$\Rightarrow \Delta Q = \dots = \frac{1}{4\pi} \int_{s_0}^{s_0+L} \Delta k \beta(s) ds$$

→ can move to a resonance!

## Chromaticity

$\frac{\Delta p}{p} \Rightarrow$  change in orbit  $\Rightarrow X_D$  (above)

$\frac{\Delta p}{p} \Rightarrow$  change in focusing.

$$k = k(p) = \underset{\substack{\uparrow \\ \text{Quad.}}}{-\frac{e}{p}} g = -\frac{e}{p_0 + \Delta p} g \approx -\frac{e}{p_0} \left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = \frac{\Delta p}{p_0} k_0$$

$$\Rightarrow \Delta Q = \frac{1}{4\pi} \frac{\Delta p}{p_0} \int k_0 \beta(s) ds$$

can move to a resonance!

Def  $\xi \equiv \frac{\Delta Q}{\Delta p/p_0} = \frac{1}{4\pi} \int k(s) \beta(s) ds$   
 $\uparrow$  Chromaticity.  $\uparrow$  All quads

How to cure chromaticity?

Cure: Focusing depending on position  
+  
Position depending on energy

A dipole - bend

A quad - focus + bend off axis.

A sextupole - ? + focus off axis + (bend  $x^2$ )

+  
Dispersion - position depending on position.

$$\begin{cases} B_x = g' x z \\ B_z = \frac{1}{2} g' (x^2 - z^2) \end{cases}$$

$$k_{\text{sext}} = \frac{e}{p} g' x = m x \quad \left( \text{compare } k_{\text{quad}} = \frac{e}{p} g \right)$$

$$= m D(s) \frac{\Delta p}{p_0}$$

Total chromaticity

$$\xi_{\text{tot}} = \frac{1}{4\pi} \oint k(s) \beta(s) ds + \frac{1}{4\pi} \oint m D(s) \beta(s) ds$$

$$= \frac{1}{4\pi} \oint (k(s) + m D(s)) \beta(s) ds$$

↑  
put  $S_x$  where there  
is dispersion!

| Problem with the case

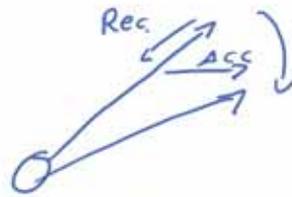
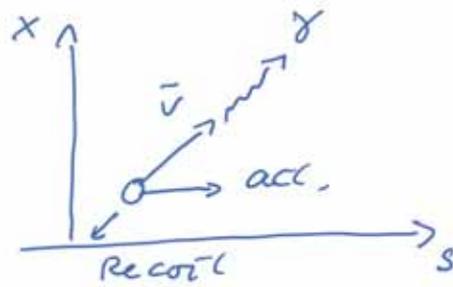
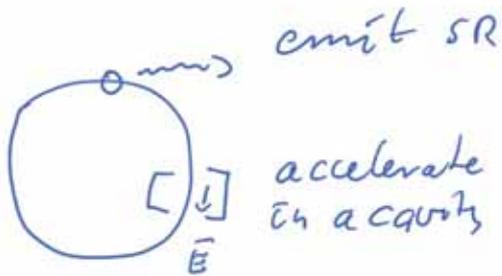
| e with  $\frac{\Delta p}{p} = 0$  but large  
| oscillation amplitudes get  
| a different focusing.

⇒ can hit a resonance.

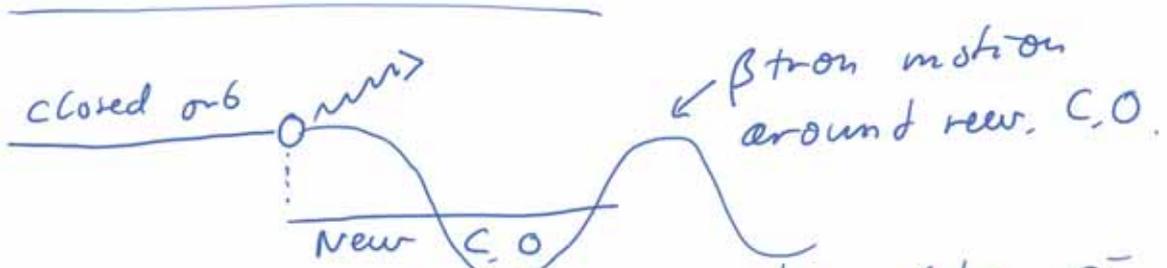
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# Damping and emittance ~~\*~~ (or, how to build a low $\epsilon$ ring)

" $\epsilon$  does not change"  
... if we do not change the energy.



$x'$  decreases  
 $\Rightarrow \epsilon$  is reduced.  
Damping.

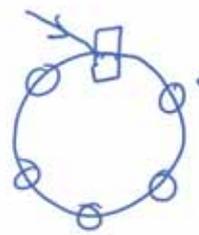
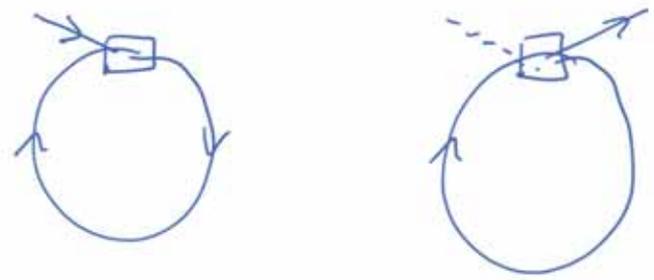


If there is dispersion when the  $e^-$  loses energy, its C.O. will change.  
 $\Rightarrow \epsilon$  growth.

- $\Rightarrow$  keep dispersion low where  $\gamma$  are emitted
- $\Rightarrow$  small dipoles
- $\Rightarrow$  many many magnets.

# Injections (for $e^-$ )

--- JUMP ---



Bunches  $\sim 300ps$   
 100-500 MHz  
 $\Rightarrow$  Difficult to switch off the magnet.



- move old beam close
- shoot in new beam

